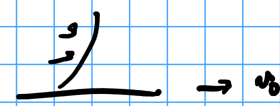
Lec 34 - Perturbation Methods & Boundary Layer theory

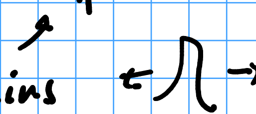
* We have learned 2 methods for solving transport PDEs
  - combination of variables



  - separation of variables
    (& method of eigenfunction expansion)

* There are a bunch of other methods!

space domain → wave# domain



linear only!
  - Fourier Transforms - for infinite domains
  - Method of characteristics - wave equations
  - Laplace Transforms - time domain → laplace/freq. domain
  - Green's functions - useful for inhomogeneous BCs & for sources & sinks.

  - Perturbation Methods - for non-linear probs

* P.M. are really important; they give us a systematic approach to NL PDEs. Else, Num. methods, or Ad hoc (trick)

* P.M. are essential to understanding Boundary Layers!

I. Perturbation Methods

A. Asymptotic Expansions

B. Regular Perturbation probs

C. Singular " "

II. Boundary layer Theory

I. Perturbation Methods

A. Asymptotic Expansions

* How start on NL probs?
* we saw for linear probs:
  - infinite series: $f(x) = \sum_{i=1}^{\infty} c_i \varphi(x)$
* Another, different kind of series.

Asymptotic series:

$$f(x) = \sum_{n=0}^{N} f_n(x)\, \varepsilon^n$$

↑ "coeff. fns"

power series in ε, small $\varepsilon \ll 1$

Taylor series:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \underbrace{\frac{f^n(x_0)}{n!}}_{\text{T.C.}} \underbrace{(x-x_0)}_{\varepsilon}{}^{n}$$

$\Rightarrow$ convergence matters as $N \to \infty$

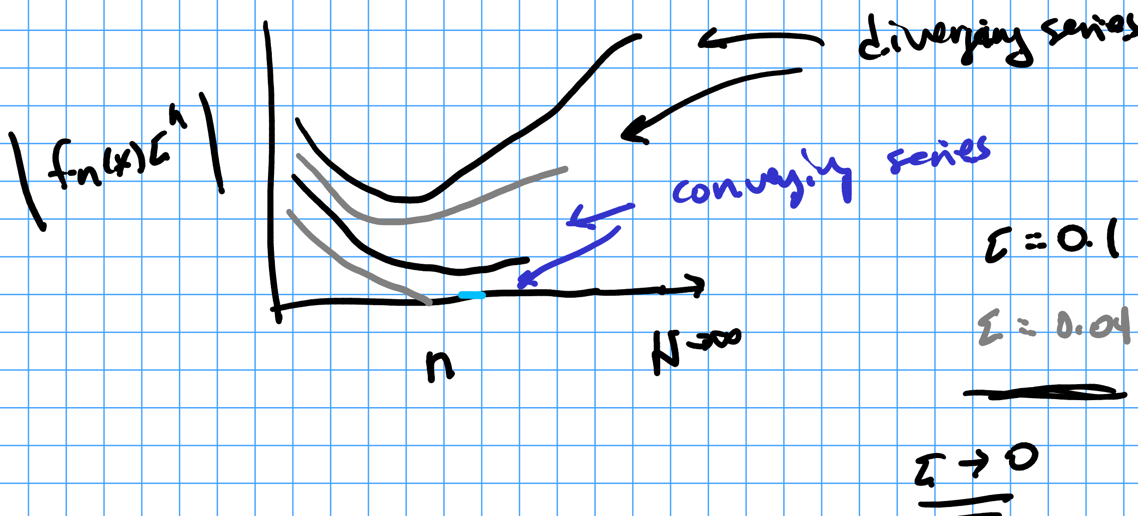
radius of convergence, R.

$\varepsilon = x - x_0$

$\varepsilon < R$

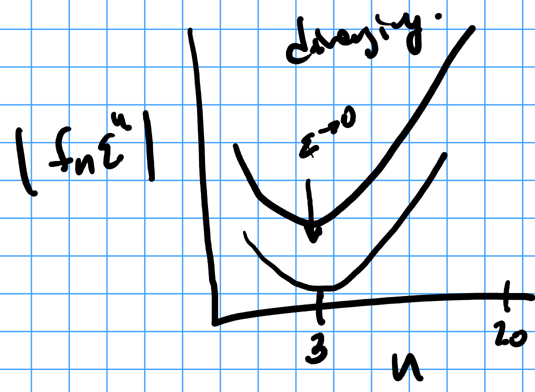
* In asymptotic series, we care about a different kind of convergence.

$$\lim_{\varepsilon \to 0} \left[\frac{f(x) - \sum_{n=0}^{N} f_n(x)\,\varepsilon^n}{\varepsilon^N} \right] = 0$$



* why is this useful
(why not just do T.S.)?

keep 20 terms!



keep 3 terms! $\varepsilon \to 0$

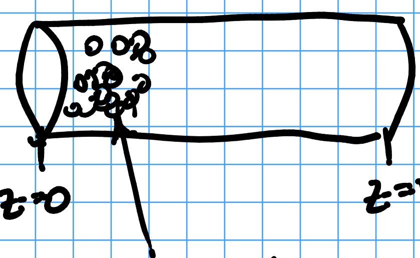
* asymptotic series is also called a "perturbation series" analysis because we are probing the changes of a small parameter, ε.

B. Regular Perturbation Problems

* Lets do an example:

consider steady-state, reaction - diffusion problem:

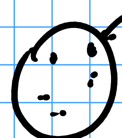
$C_A = C_{A0}$



$C_A = 0$

$z=0$ $z=L$

packed bed.



catalyst

$2A \rightarrow B$

$R_A = -kC_A^2$

$$0 = D\frac{d^2 C_A}{dz^2} - kC_A^2 \quad \leftarrow \text{from a species mass balance}$$

$$C_A(0) = C_{A0} \qquad C_A(L) = 0$$

* Non-dimensionalize:

let $\phi = \dfrac{C_A}{C_{A0}}$ $\qquad x = \dfrac{z}{L}$

$$\frac{DC_{A0}}{L^2}\frac{d^2\phi}{dx^2} = kC_{A0}^2\,\phi^2$$

$$\frac{d^2\phi}{dx^2} = \frac{L^2}{D\cancel{C_{A0}}} kC_{A0}^{\cancel{2}} \phi^2 = \frac{kC_{A0}L^2}{D} \phi^2$$

$$\boxed{\frac{d^2\phi}{dx^2} = Da\,\phi^2 \qquad \phi(0)=1,\ \phi(1)=0}$$

probably can solve by integration by parts.

* Suppose. $Da \ll 1$

$$\frac{kC_{A0}L^2}{D} \ll 1$$

$$\boxed{\frac{d^2\phi}{dx^2} = \varepsilon\phi^2 \qquad \phi(0)=1,\quad \phi(1)=0,\quad \varepsilon \ll 1 \qquad (*)}$$

* write $\phi(x)$ as an asymptotic series:

$$\phi(x) = \sum_{n=0}^{N} \varepsilon^n \phi_n(x)$$

$$= \varepsilon^0\phi_0 + \varepsilon^1\phi_1 + \varepsilon^2\phi_2 + \mathcal{O}(\varepsilon^3)$$

$$= \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \mathcal{O}(\varepsilon^3)$$

* substitute into $(*)$:

$$\frac{d^2}{dx^2}\left[\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \mathcal{O}(\varepsilon^3)\right] = \varepsilon\left[\left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \mathcal{O}(\varepsilon^3)\right)\left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \mathcal{O}(\varepsilon^3)\right)\right]$$

RHS: $= \underline{\varepsilon\phi_0^2} + \underline{\varepsilon^2\phi_0\phi_1} + \underline{\varepsilon^3}\phi_0\phi_2 + \underline{\varepsilon^2\phi_1\phi_0} + \underline{\varepsilon^3}\phi_1\phi_1 + \underline{\varepsilon^3}\phi_2\phi_0 + \mathcal{O}(\varepsilon^4)$

$$\frac{d^2\phi_0}{dx^2} + \varepsilon \frac{d^2\phi_1}{dx^2} + \varepsilon^2 \frac{d^2\phi_2}{dx^2} = \varepsilon \phi_0^2 + 2\varepsilon^2 \phi_0\phi_1$$

* since ε is arbitrary...

$O(\varepsilon^0)$: $\quad \frac{d^2\phi_0}{dx^2} = 0 \qquad \phi_0(0)=1 \,,\; \phi_0(1)=0$

$O(\varepsilon^1)$: $\quad \cancel{\varepsilon}\frac{d^2\phi_1}{dx^2} = \cancel{\varepsilon}\phi_0^2 \Rightarrow \frac{d^2\phi_1}{dx^2} = \phi_0^2 \,, \quad \phi_1(0)=0$

$\phi_1(1)=0$

$O(\varepsilon^2)$: $\quad \frac{d^2\phi_2}{dx^2} = 2\phi_0\phi_1 \qquad \phi_2(0)=0 \,,\; \phi_2(1)=0$

* 1 ODE $\Rightarrow$ 3 ODEs

NL, can't solve. $\rightarrow$ linear, can solve!

* Add BC's:

$\phi(0)=1 \Rightarrow \phi_0(0) + \varepsilon\phi_1(0) + \varepsilon^2\phi_2(0) + O(\varepsilon^3) = 1$

$\phi(1)=0 \Rightarrow \phi_0(1) + \varepsilon\phi_1(1) + \varepsilon^2\phi_2(1) + O(\varepsilon^3) = 0 \qquad \overline{\varepsilon, \varepsilon^2}$

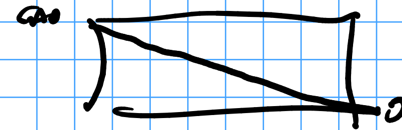
lets solve it!

$O(\varepsilon^0)$: $\quad \frac{d^2\phi_0}{dx^2} = 0 \qquad \phi_0(0)=1 \quad \phi_0(1)=0$

$\phi_0 = c_1 x + c_2 \qquad \phi_0(0) = c_2 = 1$

$\phi_0(1) = c_1 + c_2 = 0 \qquad c_1 = -1$

$$\boxed{\phi_0 = 1 - x}$$



$O(\varepsilon^1)$:

$$\frac{d^2\phi_1}{dx^2} = \phi_0^2 \qquad \phi_1(0) = 0 \qquad \phi_1(1) = 0$$

$$\frac{d^2\phi_1}{dx^2} = (1-x)^2 \Rightarrow \frac{d\phi_1}{dx} = -\frac{1}{3}(1-x)^3 + c_1$$

$$\phi_1 = +\frac{1}{3}\cdot\frac{1}{4}(1-x)^4 + c_1 x + c_2$$

$$= \frac{1}{12}(1-x)^4 + c_1 x + c_2$$

$$\phi_1(0) = \frac{1}{12} + c_2 = 0 \qquad c_2 = -1/12$$

$$\phi_1(1) = c_1 + c_2 = 0 \qquad c_1 = -c_2 = 1/12$$

$$\phi_1 = \frac{1}{12}(1-x)^4 + \frac{1}{12}x - 1/12$$

$$= \frac{1}{12}\left[(1-x)^4 - (1-x)\right]$$

$O(\varepsilon^2)$: keep going ...

$$\phi_2 = \frac{1}{504}\left[2(1-x)^7 - 7(1-x)^4 + 5(1-x)\right]$$

★ Put all together!

$$f(x) = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

$$\phi(x) = (1-x) + \frac{\varepsilon}{12}\left[(1-x)^4 - (1-x)\right]$$
$$+ \frac{\varepsilon^2}{504}\left[2(1-x)^7 - 7(1-x)^4 + 5(1-x)\right]$$

* Why did we do this?

- we can get a solution! By hand!
- Answers give physical insight!

  $\varepsilon^0 \to$ no rxn.
  $\varepsilon^1 \to$ little bit or rxn.

- Practical situations $\to$ BL theory.

  $Re \to \infty \Rightarrow \varepsilon \sim 1/Re \qquad Re \approx 10^5$

- Asymptotics to check numerical!
- Foundation for many theories
- "Abuse" this method $\varepsilon = O(1)$
- Rational approach $\to$ not just ad hoc / trick.

C. Singular Perturbation Methods