Perturbation Methods & Boundary Layer Theory

I. Perturbation Methods

A. Asymptotic Expansions

B. Regular Perturbation Problems

C. Singular Perturbation Problems

D. Summary

II. Boundary Layer Theory

we are here.

A. External Flows at high Re

B. Boundary Layer Equations

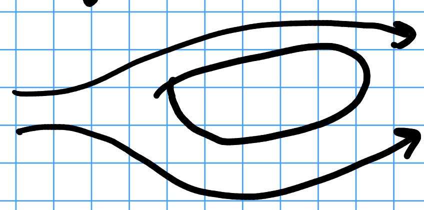
C. Blasius Solution

⋮

II. Boundary Layer Theory

A. External Flows at high Re

* "Boundary Layer Problem": Forced convection around an object.

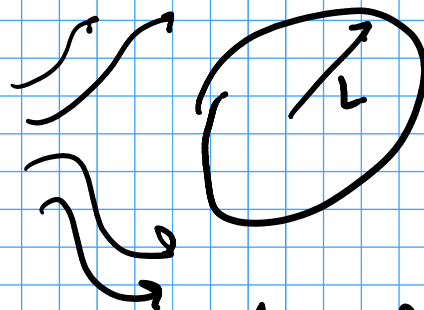


- momentum, heat, mass → all!

* we're going to focus on momentum. It's the foundation to BLPs.

$$\frac{Dc_A}{Dt} \rightarrow \frac{\partial c_A}{\partial t} + \underline{v} \cdot \underline{\nabla} c_A$$

Example:



$\underline{v}_\infty$: velocity far away

L: char size of object.

$\underline{v}_\infty,\ \rho,\ \mu \leftarrow$ constants

Eqs:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla} p + \mu\nabla^2\underline{v}$$

$$\underline{\nabla}\cdot\underline{v} = 0$$

$\underline{v} = 0$ surface (no slip)

$\underline{v} = \underline{v}_\infty$ at $r \to \infty$ $\left(v_\infty = |\underline{v}_\infty|\right)$

Non-dimensionalize

$$\underline{\tilde{x}} = \underline{x}/L \qquad \underline{\tilde{\nabla}} = L\underline{\nabla} \qquad \underline{\tilde{v}} = \underline{v}/v_\infty$$

$$\tilde{p} = p/\rho v_\infty^2 \qquad \tilde{t} = \frac{t\,v_\infty}{L}$$

$\hookrightarrow$ cont: $\underline{\tilde{\nabla}}\cdot\underline{\tilde{v}} = 0$

mom:
$$\frac{\rho v_\infty^2}{L}\frac{\partial\underline{\tilde{v}}}{\partial\tilde{t}} + \rho\frac{v_\infty^2}{L}\,\underline{\tilde{v}}\cdot\underline{\tilde{\nabla}}\,\underline{\tilde{v}} =$$

$$-\frac{1}{L}\rho v_\infty^2\,\underline{\tilde{\nabla}}\tilde{p} + \frac{\mu}{L^2}v_\infty\,\tilde{\nabla}^2\underline{\tilde{v}}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{\underline{v}} \cdot \tilde{\underline{\nabla}} \tilde{\underline{v}} = -\tilde{\nabla} \tilde{p} + \underbrace{\frac{\cancel{L}}{\rho v_\infty^{\cancel{2}}} \frac{\mu \cancel{v_\infty}}{L^{\cancel{2}}}}_{\frac{1}{Re_L}} \tilde{\nabla}^2 \tilde{\underline{v}}$$

$$\boxed{\begin{aligned} &\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\underline{\nabla}} p + \frac{1}{Re_L} \tilde{\nabla}^2 \tilde{\underline{v}} \\ &\tilde{\underline{\nabla}} \cdot \tilde{\underline{v}} = 0 \qquad \text{BC's:} \quad \tilde{v} = 0 \quad \text{surface} \\ &\qquad\qquad\qquad\qquad\quad\ \ \tilde{v} = 1 \quad \text{far away} \end{aligned}}$$

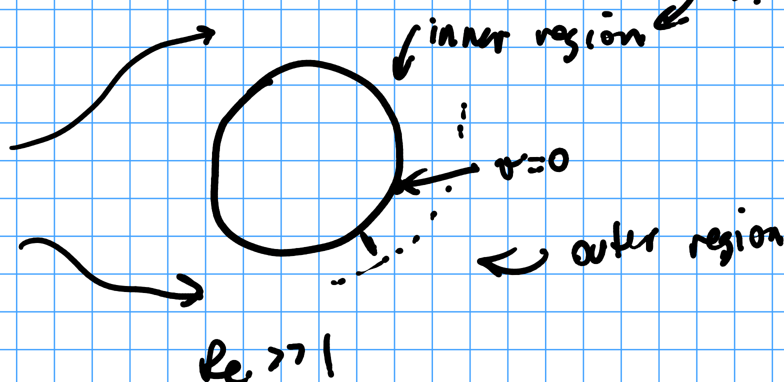
* BLP, we assume : $Re_L \gg 1$

$$\frac{1}{Re_L} = \varepsilon \ll 1$$

$$\underline{v} = \varepsilon^0 \underline{v}_0 + \varepsilon^1 \underline{v}_1 + \cdots$$

at $O(\varepsilon)$: $$\underbrace{\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\nabla} P}_{\text{Euler's Equation}} , \quad \tilde{\underline{\nabla}} \cdot \tilde{\underline{v}} = 0$$

?? **Rescale!**



* Euler's Equation

* we need to rescale using the size of the boundary layer, δ. But we don't know what this size is yet!

$\boxed{\text{Aside}}$ How do we solve Euler's Equation?

shortcut: Inviscid flow

viscous forces generate vorticity.

$\underline{\omega} = \underline{\nabla} \times \underline{v}$

Kelvin's theorem.

$\underline{\omega} = \underline{\nabla} \times \underline{v} = \underline{0}$

$\Downarrow$

$\underline{v} = \underline{\nabla}\phi$ ← velocity potential. (math: Helmholtz decomposition theorem)

+

$\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \underline{\nabla} \cdot (\underline{\nabla}\phi) = 0$

$$\boxed{\nabla^2 \phi = 0}$$

Laplace Equation
Linear, 2nd order PDE

$\phi \to \underline{v} \to p$

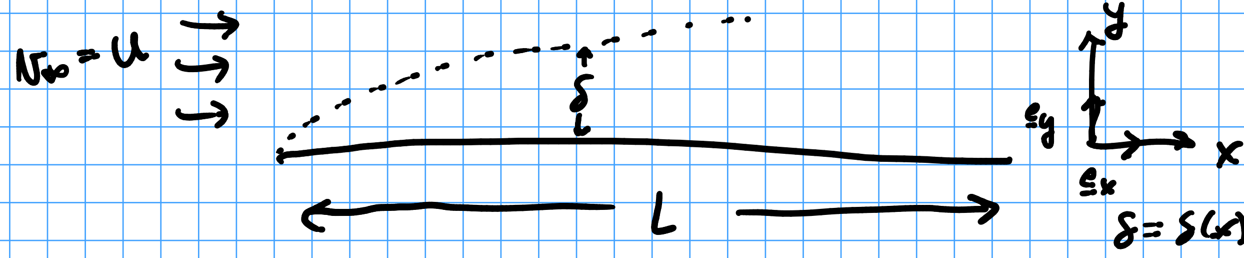
B. Boundary Layer Equations

(rescaled momentum equations)

* Now look at the inner problem.

We need a new length scale, δ ← size of the BL. (inner region)

* To be specific, let's look at a flat plate



* outer solution is trivial $\quad \underline{v} = U\, \underline{e}_x$

* Inner problem for a flat plate is almost the same for all geometries (one term different) because curvature doesn't matter. (Gary Leal, Advanced transport phenomena, §(0 F))

* Let's do rescaling (dim analysis)

$$\tilde{x} = x/L \qquad \tilde{v}_x = v_x/U$$

$$\tilde{y} = y/\delta \qquad \tilde{v}_y = v_y/V \qquad \tilde{p} = p/\rho U^2$$

(2D flat plate)

Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

find V in terms of δ.

$$\frac{U}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0 \Rightarrow \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{LV}{\delta U}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

$$\frac{LV}{\delta U} = 1 \qquad V = \frac{\delta U}{L} = \frac{\delta}{L} U$$