

Perturbation Methods & Boundary Layer Theory

I. Perturbation Methods

- A. Asymptotic Expansions
- B. Regular Perturbation Problems
- C. Singular Perturbation Problems
- D. Summary

II. Boundary Layer Theory

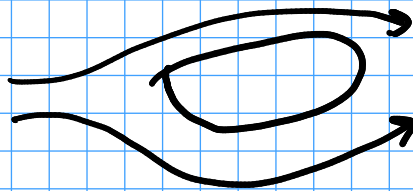
- A. External Flows at high Re
- B. Boundary Layer Equations
- C. Blasius Solution
- D. Comments

we
are
here.

III. Boundary Layer Theory

A. External Flows at high Re

* "Boundary Layer Problem": Forced convection around an object.

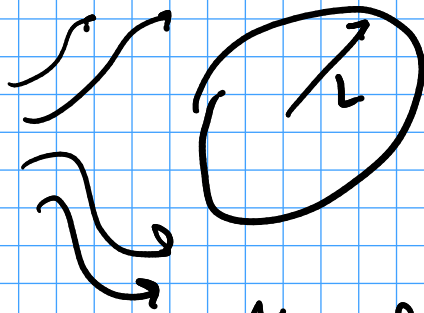


- momentum, heat, mass $\Rightarrow$ all!

* we're going to focus on momentum. It's the foundation to BLPs.

$$\frac{Dc_A}{Dt} \rightarrow \frac{\partial c_A}{\partial t} + \underline{u} \cdot \underline{\nabla} c_A$$

Example:



$\underline{v}_\infty$: velocity far away

L : char size of object.

$\underline{v}_\infty, \rho, \mu \leftarrow \text{constants}$

Eqs:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v}$$

$$\nabla \cdot \underline{v} = 0$$

$\underline{v} = 0$ surface (no slip)

$\underline{v} = \underline{v}_\infty$ at $r \rightarrow \infty$ ($v_\infty = |\underline{v}_\infty|$)

Non-dimensionalize

$$\tilde{x} = x/L \quad \tilde{\nabla} = L \nabla \quad \tilde{v} = \underline{v}/v_\infty$$

$$\tilde{p} = p/\rho v_\infty^2 \quad \tilde{t} = \frac{t v_\infty}{L}$$

cont: $\tilde{\nabla} \cdot \tilde{v} = 0$

mom:

$$\frac{\rho v_\infty^2}{L} \frac{\partial \tilde{v}}{\partial \tilde{t}} + \rho \frac{v_\infty^2}{L} \tilde{v} \cdot \tilde{\nabla} \tilde{v} =$$

$$-\frac{1}{L} \rho v_\infty^2 \tilde{\nabla} \tilde{p} + \frac{\mu}{L^2} \tilde{\nabla}^2 \tilde{v}$$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \tilde{\nabla} \tilde{u} = -\tilde{\nabla} p + \underbrace{\frac{\mu \nu}{\rho \nu L^2}}_{\frac{1}{Re_L}} \tilde{\nabla}^2 \tilde{u}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= -\tilde{\nabla} p + \frac{1}{Re_L} \tilde{\nabla}^2 \tilde{u} \\ \tilde{\nabla} \cdot \tilde{u} &= 0 \end{aligned}} \quad \text{BC's:} \quad \begin{array}{ll} \tilde{u} = 0 & \text{surface} \\ \tilde{u} = 1 & \text{far away} \end{array}$$

* BLP, we assume: $Re_L \gg 1$

$$\frac{1}{Re_L} = \varepsilon \ll 1$$

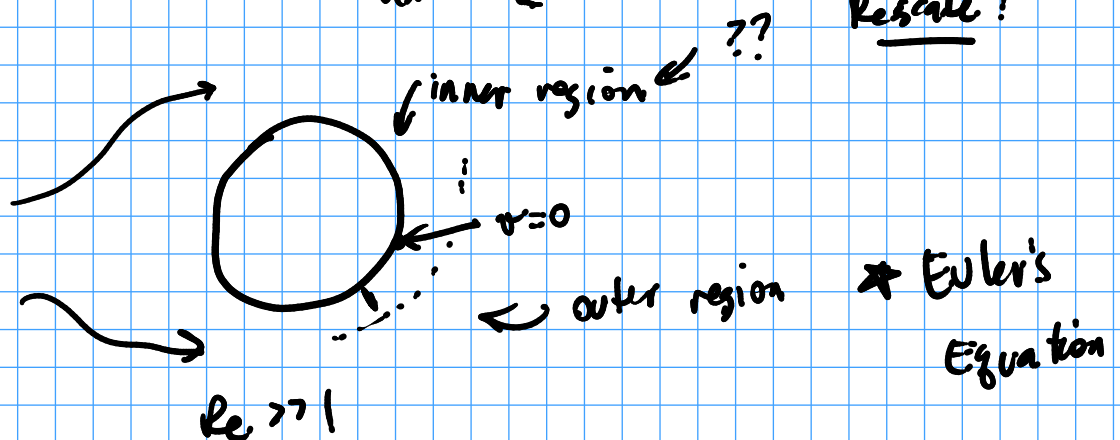
$$\tilde{u} = \varepsilon^0 u_0 + \varepsilon^1 u_1 + \dots$$

at $\mathcal{O}(\varepsilon)$:

$$\underbrace{\frac{\partial \tilde{u}}{\partial t} = -\tilde{\nabla} p, \quad \tilde{\nabla} \cdot \tilde{u} = 0}_{\text{Euler's Equation}}$$

Euler's Equation

Rescale!



* we need to rescale using the size of the boundary layer, δ . But we don't know what this size is yet!

Aside How do we solve Euler's Equation?

Shortcut: Inviscid flow viscous forces generate vorticity.

Kelvin's theorem.

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \underline{0}$$

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$

$\downarrow$
 $\underline{u} = \underline{\nabla} \phi \leftarrow$ velocity potential. (math:

+

$$\underline{\nabla} \cdot \underline{u} = 0 \Rightarrow \underline{\nabla} \cdot (\underline{\nabla} \phi) = 0$$

Helmholtz decomposition theorem)

$$\boxed{\nabla^2 \phi = 0}$$

Laplace Equation
Linear, 2nd order PDE

$$\phi \rightarrow \underline{u} \rightarrow p$$

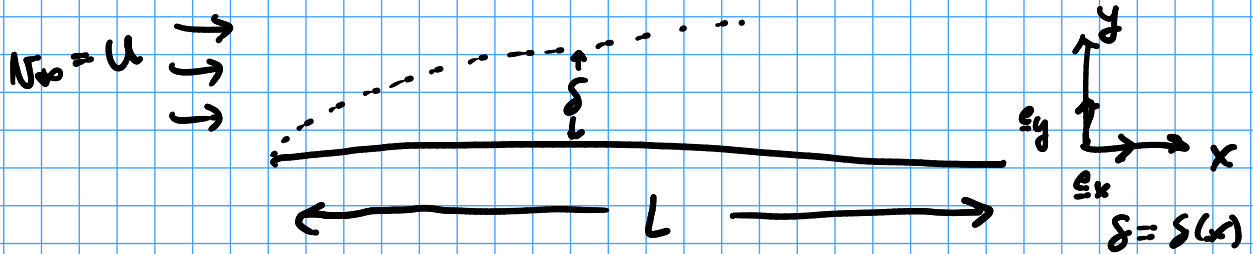
B. Boundary Layer Equations

(rescaled momentum equations)

* Now look at the inner problem.

We need a new length scale, $\delta \leftarrow$ size of the bl.
(inner region)

* To be specific, let's look at a flat plate



* outer solution is trivial $\underline{v} = U \underline{e}_x$

* Inner problem for a flat plate

is almost the same for all geometries

(one term different) because curvature

doesn't matter. (Lang leal, Advanced transport

phenomena, §10.4)

* Let's do rescaling (dim analysis)

$$\tilde{x} = x/L$$

$$\tilde{u}_x = u_x/U$$

$$\tilde{y} = y/\delta$$

$$\tilde{u}_y = u_y/\nu$$

$$\tilde{p} = p/\rho U^2$$

(2D flat plate)

Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{U}{L} \frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \frac{V}{\delta} \frac{\partial \tilde{u}_y}{\partial \tilde{y}} = 0 \Rightarrow \frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \frac{LV}{\delta U} \frac{\partial \tilde{u}_y}{\partial \tilde{y}} = 0$$

find V in
terms of δ .

$$\frac{LV}{\delta U} = 1$$

$$V = \frac{\delta U}{L} = \frac{\delta}{L} U$$

x-momentum (2D)

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

steady

$$\rho u \tilde{v}_x \cdot \frac{u}{L} \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \rho v \tilde{v}_y \frac{u}{\delta} \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = \frac{\rho u^2}{L} \left(-\frac{\partial \tilde{p}}{\partial \tilde{x}} \right)$$

$$+ \mu \frac{u}{L^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \mu \frac{u}{\delta^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\frac{\rho u^2}{L} \tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\rho v u}{\delta} \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\rho u^2}{L} \frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+ \frac{\mu u}{L^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\mu u}{\delta^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{y}} + \frac{L}{\rho u^2} \frac{\rho u u}{\delta} \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

$$+ \frac{L}{\rho u^2} \cdot \frac{\mu u}{L^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{L}{\rho u^2} \frac{\mu u}{\delta^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\frac{1}{Re_L}$$

$$\frac{\mu}{\rho u L} \frac{L^2}{\delta^2} = \frac{1}{Re_L} \cdot \left(\frac{L}{\delta} \right)^2$$

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re_L} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{1}{Re_L} \frac{L^2}{\delta^2} \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$1/Re_L = \varepsilon \ll 1$$

$$\frac{1}{Re_L} \frac{L^2}{\delta^2} = 1$$

$$\frac{\delta^2}{L^2} = Re_L^{-1} \Rightarrow \frac{\delta}{L} = Re_L^{-1/2} = \varepsilon^{1/2}$$

At $O(\varepsilon)$:

$$\tilde{u}_x \frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \tilde{u}_y \frac{\partial \tilde{u}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}_x}{\partial \tilde{y}^2}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

w/units again

BLE,
for the
inner
region

$$\delta \propto L \cdot Re_L^{-1/2} \propto L \cdot \left(\frac{\mu}{\rho u L} \right)^{1/2} \propto \left(\frac{L \mu}{\rho u} \right)^{1/2}$$

$$\delta \propto \left(\frac{L \nu}{u} \right)^{1/2}$$

BL height!

$$\delta = C \cdot \left(\frac{L \nu}{u} \right)^{1/2}$$

* What about u_y ? $\rightarrow$ continuity Equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u_y = - \int \frac{\partial u_x}{\partial x} dy$$

+ y-momentum:

$$\frac{\partial p}{\partial y} = 0$$

C. Blasius Solution to the BLEs

* Summary of the BL problem

(outer) $\nabla^2 \phi = 0$ $\underline{v} = \nabla \phi$ $\underline{v}(r \rightarrow \infty) = \underline{v}_\infty$

(inner) $u_x \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$

$\frac{\partial p}{\partial y} = 0$ $v_y = \int \frac{\partial u_x}{\partial x} dy$

* Steps to solving a BL problem $\underline{v}(r=0) = 0$

(1) Solve outer problem, Use outer BC

$\Rightarrow \underline{v}_{\text{outer}}$

(2) Find $\frac{\partial p}{\partial x}$ using Euler's Equation

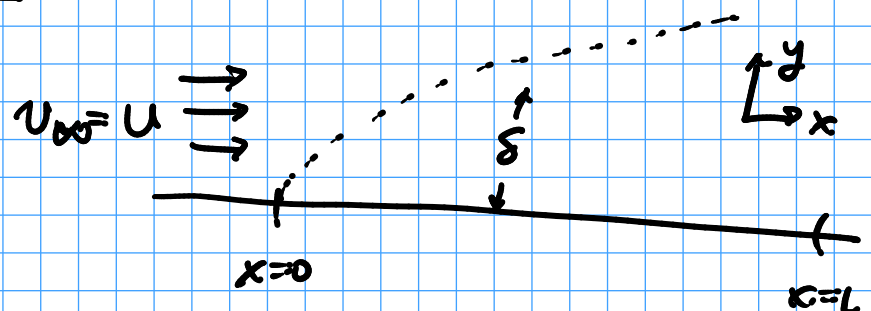
$$\rho \frac{D\underline{v}}{Dt} = -\nabla p$$

(3) Solve the BLEs (inner region)
for u_x, v_y

(4) Use inner BC & matching condition
to find the constants.

1. Blasius Equation

Ex: Flat Plate



(1) The outer solution.

$$u_x = u \quad (\text{trivial!})$$

$$u_y = 0$$

(2) Pressure gradients

$$\left. \begin{aligned} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \end{aligned} \right\} \text{Euler's Equation}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

← will be $\neq 0$
for curved
objects
(remember
Bernoulli!)

(3) The PL Equations

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial x} = 0$$

* Not trivial!

* Non-linear, PDE

↓
Let's try a similarity transform
(comb. of vars)

$$\eta = \frac{y}{\delta(x)} \quad \delta(x) = \left(2 \frac{\nu x}{u} \right)^{1/2}$$

↓
ODE fn of η
PDE fn of x, y

Aside Stream Functions

* Streamfunction $\Rightarrow$ allow you to simultaneously satisfy continuity

$$\underline{v} = \nabla \times \underline{\Psi}$$

$$\underline{\Psi} = [0, 0, -\Psi]$$

↑ capital

↑ lowercase

$$\underline{v} = -\frac{\partial \Psi}{\partial y} \underline{e}_x + \frac{\partial \Psi}{\partial x} \underline{e}_y$$

$$v_x = -\frac{\partial \Psi}{\partial y}$$

$$v_y = \frac{\partial \Psi}{\partial x}$$

$$\nabla \cdot \underline{v} = \nabla \cdot (\nabla \times \underline{\Psi}) = 0$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial x} \left(-\frac{\partial \Psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \Psi}{\partial x} \right)$$

$$= 0$$

* Only good for 2D (or axisymmetric)

- Auto satisfies continuity $\phi, \theta \rightarrow r, \theta$

- 1 variable instead of 2.