Perturbation Methods & Boundary Layer Theory

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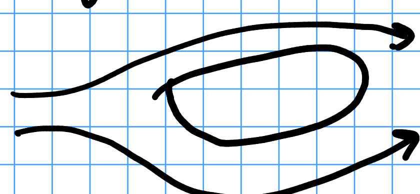
we are here.

II. Boundary Layer Theory

A. External Flows at high Re

* "Boundary Layer Problem": Forced convection around an object.

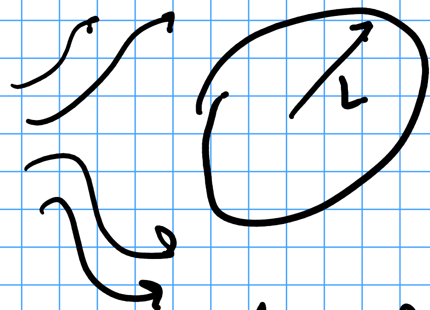


- momentum, heat, mass → all!

* we're going to focus on momentum. It's the foundation to BLPs.

$$\frac{Dc_A}{Dt} \to \frac{\partial c_A}{\partial t} + \underline{v} \cdot \nabla c_A$$

Example:



$\underline{v}_\infty$: velocity far away

L: char size of object.

$\underline{v}_\infty, \rho, \mu \leftarrow$ constants

Eqs:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla} p + \mu\nabla^2 \underline{v}$$

$$\underline{\nabla}\cdot\underline{v} = 0$$

$\underline{v} = 0$ surface (no slip)

$\underline{v} = \underline{v}_\infty$ at $r \to \infty$ $\quad (v_\infty = |\underline{v}_\infty|)$

Non-dimensionalize

$$\tilde{x} = x/L \qquad \tilde{\underline{\nabla}} = L\underline{\nabla} \qquad \tilde{\underline{v}} = \underline{v}/v_\infty$$

$$\tilde{p} = p/\rho v_\infty^2 \qquad \tilde{t} = \frac{t\, v_\infty}{L}$$

cont: $\tilde{\underline{\nabla}}\cdot\tilde{\underline{v}} = 0$

mom:
$$\frac{\rho v_\infty^2}{L}\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}} + \rho\frac{v_\infty^2}{L}\,\tilde{\underline{v}}\cdot\tilde{\underline{\nabla}}\,\tilde{\underline{v}} =$$

$$-\frac{1}{L}\rho v_\infty^2\,\tilde{\underline{\nabla}}\tilde{p} + \frac{\mu}{L^2}v_\infty\,\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}} + \tilde{\underline{v}} \cdot \tilde{\underline{\nabla}} \tilde{\underline{v}} = -\tilde{\nabla} \tilde{p} + \underbrace{\frac{\cancel{L}}{\rho v_\infty^{\cancel{2}}} \frac{\mu \cancel{v_\infty}}{L^{\cancel{2}}}}_{\frac{1}{Re_L}} \tilde{\nabla}^2 \tilde{\underline{v}}$$

$$\boxed{\begin{aligned} &\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\underline{\nabla}} p + \frac{1}{Re_L} \tilde{\nabla}^2 \tilde{\underline{v}} \\ &\tilde{\underline{\nabla}} \cdot \tilde{\underline{v}} = 0 \qquad \text{BC's: } \quad \tilde{v} = 0 \quad \text{surface} \\ &\qquad\qquad\qquad\qquad\quad\; \tilde{v} = 1 \quad \text{far away} \end{aligned}}$$

* BLP, we assume : $Re_L \gg 1$

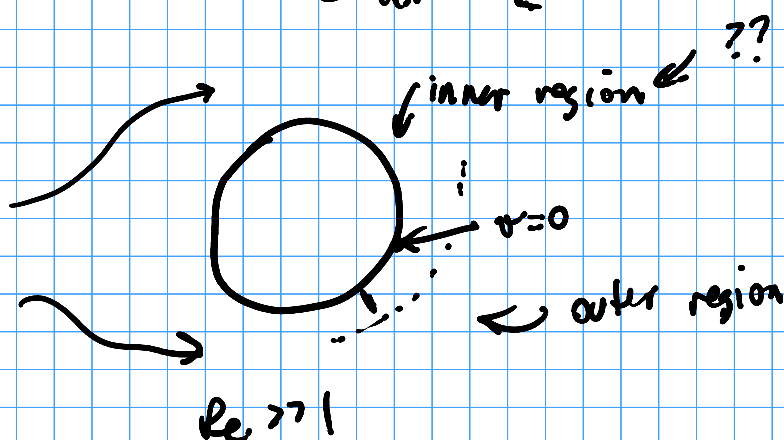
$$\frac{1}{Re_L} = \varepsilon \ll 1$$

$$\underline{v} = \varepsilon^0 \underline{v}_0 + \varepsilon^1 \underline{v}_1 + \dots$$

at $O(\varepsilon)$:

$$\underbrace{\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\nabla} P}_{\text{Euler's Equation}} , \quad \tilde{\underline{\nabla}} \cdot \tilde{\underline{v}} = 0$$

Rescale !



* Euler's Equation

* we need to <u>rescale</u> using the size of the boundary layer, δ. But we don't know what this size is yet!

$\boxed{\text{Aside}}$ How do we solve Euler's Equation?

shortcut: Inviscid flow — viscous forces generate <u>vorticity</u>.

$\underline{\omega} = \underline{\nabla} \times \underline{v}$

Kelvin's theorem.

$\underline{\omega} = \underline{\nabla} \times \underline{v} = \underline{0}$

$\Downarrow$

$\underline{v} = \underline{\nabla}\phi$ ← velocity potential. (math: Helmholtz decomposition theorem)

+

$\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \underline{\nabla} \cdot (\underline{\nabla}\phi) = 0$

$$\boxed{\nabla^2 \phi = 0}$$

Laplace Equation
Linear, 2nd order PDE

$\phi \rightarrow \underline{v} \rightarrow p$

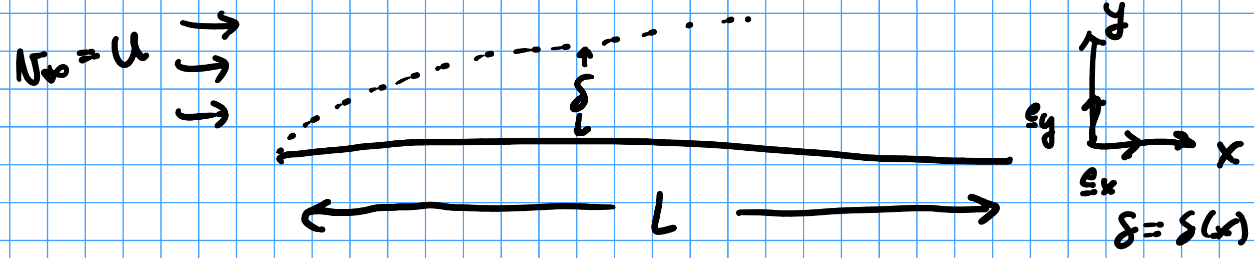
B. Boundary Layer Equations

(rescaled momentum equations)

* Now look at the inner problem.

We need a new length scale, δ ← size of the BL. (inner region)

* To be specific, let's look at a flat plate



* outer solution is trivial $\quad \underline{v} = U \underline{e}_x$
* Inner problem for a flat plate is almost the same for all geometries (one term different) because curvature doesn't matter. (Gary Leal, Advanced transport phenomena, §10 F)

* Let's do rescaling (dim analysis)

$$\tilde{x} = x/L \qquad \tilde{v}_x = v_x/U$$

$$\tilde{y} = y/\delta \qquad \tilde{v}_y = v_y/V \qquad \tilde{p} = p/\rho U^2$$

(2D flat plate)

Continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

find V in terms of δ.

$$\frac{U}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0 \Rightarrow \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{LV}{\delta U}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

$$\frac{LV}{\delta U} = 1 \qquad V = \frac{\delta U}{L} = \frac{\delta}{L} U$$

x-momentum (2D)

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

steady

$$\rho U \tilde{v}_x \cdot \frac{U}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \rho V \tilde{v}_y \frac{U}{\delta}\frac{\partial \tilde{v}_x}{\partial \tilde{y}} = \frac{\rho U^2}{L}\left(-\frac{\partial \tilde{P}}{\partial \tilde{x}}\right) + \mu\frac{U}{L^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \mu\frac{U}{\delta^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\frac{\rho U^2}{L}\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\rho V U}{\delta}\tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\rho U^2}{L}\frac{\partial \tilde{P}}{\partial \tilde{x}} + \frac{\mu U}{L^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\mu U}{\delta^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{y}} + \underline{\frac{\cancel{L}}{\cancel{\rho U^2}}\frac{\cancel{\rho U}U\delta}{\delta\cancel{L}}}\tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \underbrace{\frac{\cancel{L}}{\rho U^2}\cdot\frac{\mu U}{L^{\cancel{2}}}}_{\frac{1}{Re_L}}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \underbrace{\frac{L}{\rho U^2}\frac{\mu U}{\delta^2}}_{\frac{\mu}{\rho U L}\frac{L^2}{\delta^2} = \frac{1}{Re_L}\cdot\left(\frac{L}{\delta}\right)^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \frac{1}{Re_L}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{1}{Re_L}\frac{L^2}{\delta^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$1/Re_L = \varepsilon \ll 1$ $\qquad \frac{1}{Re_L}\frac{L^2}{\delta^2} = 1$

$$\frac{\delta^2}{L^2} = Re_L^{-1} \Rightarrow \frac{\delta}{L} = Re_L^{-1/2}$$
$$= \varepsilon^{1/2}$$

At $\mathcal{O}(\varepsilon^0)$:

$$\boxed{\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}}$$

w/ units again

$$\boxed{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}}$$

BLE, for the inner region

$$\delta \propto L \cdot Re_L^{-1/2} \propto L \cdot \left(\frac{\mu}{\rho u L}\right)^{1/2} \propto \left(\frac{L\mu}{\rho u}\right)^{1/2}$$

$$\delta \propto \left(\frac{L\nu}{u}\right)^{1/2}$$

BL height!

$$\boxed{\delta = c \cdot \left(\frac{L\nu}{u}\right)^{1/2}}$$

* What about v_y? $\rightarrow$ continuity Equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\boxed{v_y = -\int \frac{\partial v_x}{\partial x}\, dy}$$

* y-momentum: $\boxed{\frac{\partial p}{\partial y} = 0}$

C. Blasius Solution to the BLEs

* Summary of the BL problem

(outer) $\nabla^2 \phi = 0$ $\quad \underline{v} = \nabla \phi$ $\quad \underline{v}(r \to \infty) = \underline{v}_\infty$

(inner) $$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0 \qquad v_y = -\int \frac{\partial v_x}{\partial x}\, dy$$

$$\underline{v}(\underline{r} = 0) = 0$$

* Steps to solving a BL problem

(1) Solve outer problem, use outer BC

$\Rightarrow \underline{v}_{outer}$

(2) Find $\frac{\partial p}{\partial x}$ using euler's equation

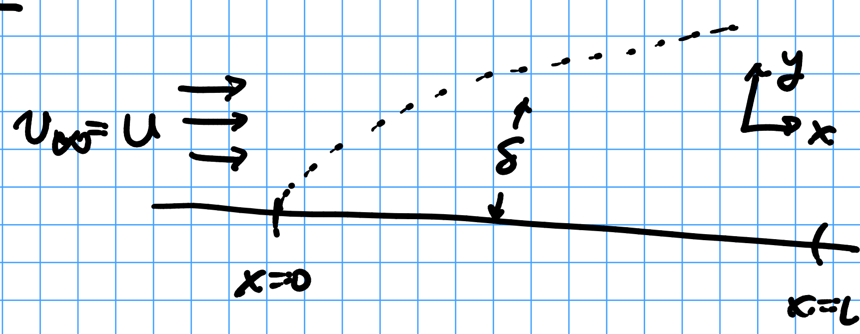
$$\rho \frac{D\underline{v}}{Dt} = -\nabla p$$

(3) Solve the BLEs (inner region) for v_x, v_y

(4) Use inner BC & matching condition to find the constants.

1. Blasius Equation

Ex: Flat Plate



(1) The outer solution.

$v_x = U$ (trivial!)

$v_y = 0$

(2) Pressure gradients

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

Euler's Equation

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$$

← will be $\neq 0$ for curved objects (remember Bernoulli)

(3) The BL Equations

$\partial P / \partial x = 0$

$$v_x \frac{\partial v_x}{\partial x} + \underset{\sim}{v_y} \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

* Not trivial!
* Non-linear, PDE

lets try a similarity transform (comb. of vars)

$\eta = y / \delta(x)$ $\quad \delta(x) = \left(\frac{2\nu x}{U}\right)^{1/2}$

↳ ODE fn of η

PDE fn of x, y

$\boxed{\text{Aside}}$ Stream Function

* streamfunction ⇒ allow you to simultaneously satisfy continuity

$\underline{v} = \underline{\nabla} \times \underline{\Psi}$ $\quad \underline{\Psi} = [0, 0, -\psi]$

capital

lowercase

$$\underline{v} = -\frac{\partial \psi}{\partial y} \underline{e}_x + \frac{\partial \psi}{\partial x} \underline{e}_y$$

$$\boxed{v_x = -\frac{\partial \psi}{\partial y}} \qquad \boxed{v_y = \frac{\partial \psi}{\partial x}}$$

$$\underline{\nabla} \cdot \underline{v} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{\Psi}) = 0$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= 0$$

* Only good for $\underline{2D}$ (or axisymmetric)

ϕ, θ or r, θ

- Auto satisfies continuity
- 1 variable instead of 2.

$\boxed{\text{End of aside}}$

* Simultaneously do stream fn & rescaling

$$\eta = y/\delta = y\Big/\sqrt{2\nu x/U}$$

$$\frac{v_x}{U} = \frac{df}{d\eta} \leftarrow f? \qquad \psi = -U\delta f(\eta)$$

dimensionless stream function...

Blasius did this

* Plug these definitions into

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\boxed{v_x = u f'}$$

$$f' = \frac{df}{d\eta}, \quad f'' = \frac{d^2 f}{d\eta^2}, \ldots$$

$$\frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x}\left(u f'\right)$$

get rid of ... $\frac{d}{d\eta}$

use chain rule.

$$= u \frac{\partial}{\partial x} f' = u \underbrace{\frac{\partial \eta}{\partial x}} \left[\frac{\partial}{\partial \eta} f'\right]$$

$$\frac{\partial}{\partial \eta}\frac{\partial f}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2} = f''$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{2\nu x/u}}\right)$$

$$= -y \frac{1}{\cancel{2}} \left(2\nu x/u\right)^{-3/2} \cdot \frac{\cancel{2}\nu}{u}$$

$\eta = y/\delta$ $\quad$ $\delta = \left(\frac{2\nu x}{u}\right)^{1/2}$

$$= -y\, \delta^{-3} \frac{\nu}{u}$$

$$= -\eta\, \delta^{-2} \nu/u$$

$$\frac{\partial v_x}{\partial x} = -\eta \delta^{-2} \frac{\nu}{u} \cdot u \cdot f''$$

$$\boxed{\frac{\partial v_x}{\partial x} = -\eta \delta^{-2} \nu f''}$$

use integration by parts

$$v_y = \frac{\nu}{\delta}\left[\eta f' - f\right]$$

$$\frac{\partial v_x}{\partial y} = \frac{u f''}{\delta} \qquad \frac{\partial^2 v_x}{\partial y^2} = \frac{u}{\delta^2} f''' \ *$$

$$\underline{uf'\left(-\eta\, \delta^{-2} \nu f''\right)} + \frac{\nu}{\delta}\left[\eta f' - f\right] \cdot \frac{uf''}{\delta} = \frac{\nu u}{\delta^2} f'''$$

-- rescaled

- written w/ dimless streamfunction

$$\cancel{\underline{-\frac{\eta u \nu}{\delta^2}\left(f'f''\right)}} + \underline{\frac{\nu u}{\delta^2}\left[\cancel{\eta f' f''}\right.} - \left. ff''\right] = \frac{\nu u}{\delta^2} f'''$$

$$-\frac{\cancel{\nu u}}{\cancel{\delta^2}} ff'' = \frac{\cancel{\nu u}}{\cancel{\delta^2}} f'''$$

$$\boxed{f''' + \underline{ff''} = 0}$$

Blasius' Equation !!

*We need BC's:

$v_x(0)=0$ (no slip) $\Rightarrow$ $\boxed{f'(0) = 0}$ inner

$v_y(0)=0$ (no penetration) $\boxed{f(0) = 0}$

$v_x(\infty) \to u$ $\quad uf'(\infty) = u$ $\rightarrow$ $\boxed{f'(\infty) = 1}$ matching

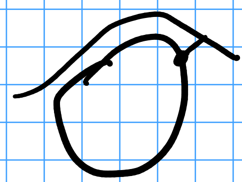
* Well posed, ODE (nonlinear!)

Need numerics to solve!

η	f	f'
⋮	⋮	⋮

D. Comments

* we solved flat plate BL problem.



$\frac{dP}{dx} \rightarrow$ separation point

* turbulence!

* these are all <u>laminar</u> BL problems.

* There's another way to solve the problems
  - → approximate solution - von karmen solution ← integrate the momentum equation
  - → like to have an explicit solution

$$\frac{v_x}{u} = \phi(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

$$(*) \quad \frac{v_x}{u} = \frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3$$

$$\delta(x) = \underbrace{\sqrt{\frac{280}{13}}}_{\text{diff const.}} \left(\frac{\nu x}{u}\right)^{1/2}$$