Vector Products in curvilinear Coordinates

what is $\underline{v} \cdot \underline{w}$ in curvilinear coordinates?

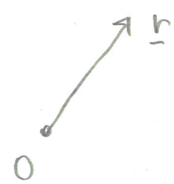
* The dot product is given by

$$\underline{v} \cdot \underline{w} = \sum_i v_i w_i$$

for any orthogonal coordinate system. A similar result holds for other products.

* Be careful. This does not apply to products of position vectors. Position vectors are given by:

$$\underline{r} = x\, \underline{e}_x + y\, \underline{e}_y + z\, \underline{e}_z$$



Example 1: Regular dot product of field variables

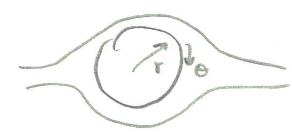
Consider the velocity field in cylindrical coordinates

$$\underline{v} = v_r\, \underline{e}_r + v_\theta\, \underline{e}_\theta$$

$$v_r = u \cos\theta \left[1 - \left(\frac{R}{r}\right)^2 \right]$$

$$v_\theta = -u \sin\theta \left[1 + \left(\frac{R}{r}\right)^2 \right]$$

Flow at high Re around a cylinder



(a) In cylindrical coordinates

$$\underline{v} \cdot \underline{v} = v_r^2 + v_\theta^2$$

$$v_r^2 = u^2 \cos^2\theta \left[1-\left(\frac{R}{r}\right)\right]^2$$

$$v_\theta^2 = u^2 \sin^2\theta \left[1+\left(R/r\right)^2\right]^2$$

$$v^2 = v_r^2 + v_\theta^2 = u^2\cos^2\theta\left[1 - 2\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right]$$

$$+ u^2 \sin^2\theta \left[1 + 2\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right]$$

$$= u^2 \left(\cos^2\theta + \sin^2\theta\right)\left[1 + \left(R/r\right)^4\right]$$

$$+ u^2 \left(\sin^2\theta - \cos^2\theta\right)(2)\left(R/r\right)^2$$

recall: $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$\boxed{\frac{v^2}{u^2} = 1 + 2\left(\frac{R}{r}\right)^2 \cos(2\theta) + \left(\frac{R}{r}\right)^4}$$

(b) In cartesian coordinates

* Convert to cartesian coordinates first

$$\left.\begin{aligned} \underline{e}_r &= \cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y \\ \underline{e}_\theta &= -\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y \end{aligned}\right\} \text{relation between unit vectors}$$

$$\underline{v} = v_x \underline{e}_x + v_y \underline{e}_y = \underbrace{v_r \underline{e}_r + v_\theta \underline{e}_\theta}_{\text{write this and sub in } \underline{e}_r \,\&\, \underline{e}_\theta}$$

$$= u\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right] \times \left(\cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y\right)$$

$$- u\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\left(-\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y\right)$$

$$= u\cos^2\theta\left[1-(R/r)^2\right]\underline{e}_x + u\cos\theta\sin\theta\left[1-(R/r)^2\right]\underline{e}_y$$

$$+ u\sin^2\theta\left[1+(R/r)^2\right]\underline{e}_x - u\cos\theta\sin\theta\left[1+(R/r)^2\right]\underline{e}_y$$

$$v_x = u\cos^2\theta\left[1-(R/r)^2\right] + u\sin^2\theta\left[1+(R/r)^2\right]$$

$$v_y = u\cos\theta\sin\theta\left\{\cancel{1} - (R/r)^2 - \cancel{1} - (R/r)^2\right]$$

⇓ simplify both

$$v_x = u\cos^2\theta + u\sin^2\theta - u\cos^2\theta(R/r)^2 + u\sin^2\theta(R/r)^2$$

$$= u\underbrace{[\cos^2\theta + \sin^2\theta]}_{1} - u\left(\frac{R}{r}\right)^2\underbrace{[\cos^2\theta - \sin^2\theta]}_{\cos 2\theta}$$

$$v_x = u - u\left(\frac{R}{r}\right)^2\cos 2\theta \quad ✓$$

$$v_y = -2u\cos\theta\sin\theta\left(\frac{R}{r}\right)^2$$

Now, compute: $\underline{v}\cdot\underline{v}$

$$v^2 = \underline{v}\cdot\underline{v} = v_x^2 + v_y^2 = u^2\left[1 - \left(\frac{R}{r}\right)^2\cos 2\theta\right]^2 + 4u^2\cos^2\theta\sin^2\theta(R/r)^4$$

$$= u^2\left[1 - 2\left(\frac{R}{r}\right)^2\cos 2\theta + \left(\frac{R}{r}\right)^4\cos^2 2\theta\right] + 4u^2\cos^2\theta\sin^2\theta(R/r)^4$$

$$\frac{v^2}{u^2} = 1 - 2\left(\frac{R}{r}\right)^2\cos 2\theta + \left(\frac{R}{r}\right)^4\left[\cos^2 2\theta + 4\cos^2\theta\sin^2\theta\right]$$

expand & simplify

$$\cos^2(2\theta) = (\cos^2\theta - \sin^2\theta)^2 = \cos^2\theta\cos^2\theta - 2\cos^2\theta\sin^2\theta + \sin^2\theta\sin^2\theta$$

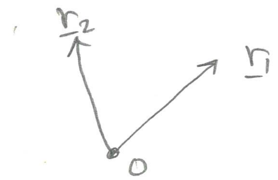
$$\cos^2(2\theta) + 4\cos^2\theta\sin^2\theta = \cos^2\theta\cos^2\theta + 2\cos^2\theta\sin^2\theta + \sin^2\theta\sin^2\theta$$

$$= (\cos^2\theta + \sin^2\theta)^2 = 1^2 = 1$$

$$\boxed{\frac{v^2}{u^2} = 1 - 2(R/r)^2\cos 2\theta + (R/r)^4}$$

Same! The formula is correct.

Example 2: Dot product of position vectors



$$\underline{r}_1 = x_1 \underline{e}_x + y_1 \underline{e}_y + z_1 \underline{e}_z$$

$$\underline{r}_2 = x_2 \underline{e}_x + y_2 \underline{e}_y + z_2 \underline{e}_z$$

Dot product: $\underline{r}_1 \cdot \underline{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$

* Convert to cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\underline{r}_1 \cdot \underline{r}_2 = r_1\cos\theta_1 \cdot r_2\cos\theta_2 + r_1\sin\theta_1 \cdot r_2\sin\theta_2 + z_1 z_2$$

$$= r_1 r_2 (\underbrace{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}_{\cos(\theta_1-\theta_2)}) + z_1 z_2$$

$\cos(\theta_1 - \theta_2)$ ← trig identity

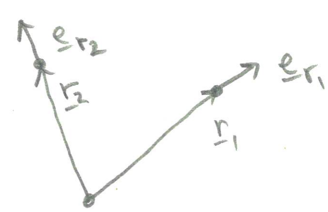
$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta_1 - \theta_2) + z_1 z_2}$$

* Compare this to the formula valid for field variables

$$\underline{r}_1 = r_1 \underline{e}_r + z_1 \underline{e}_z \qquad \underline{r}_2 = r_2 \underline{e}_r + z_2 \underline{e}_z$$

$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 + z_1 z_2} \quad \times \text{ wrong!}$$

why? Because $\underline{e}_r$ for $\underline{r}_1$ is not the same as $\underline{e}_r$ for $\underline{r}_2$:



$$\underline{e}_r = \cos\theta \, \underline{e}_x + \sin\theta \, \underline{e}_y$$

* Let's try this instead!

$$\underline{r}_1 \cdot \underline{r}_2 = (r_1 \, \underline{e}_{r_1}) \cdot (r_2 \, \underline{e}_{r_2}) + z_1 \underline{e}_{z_1} \cdot z_2 \underline{e}_{z_2}$$

$$= r_1 r_2 \, \underbrace{(\underline{e}_{r_1} \cdot \underline{e}_{r_2})} + (z_1 z_2) \underbrace{(\underline{e}_{z_1} \cdot \underline{e}_{z_2})}_{1}$$

$$\underline{e}_{r_1} \cdot \underline{e}_{r_2} = (\cos\theta_1 \, \underline{e}_x + \sin\theta_1 \, \underline{e}_y) \cdot (\cos\theta_2 \, \underline{e}_x + \sin\theta_2 \, \underline{e}_y)$$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

using $\underline{e}_x \cdot \underline{e}_x = 1$, $\underline{e}_y \cdot \underline{e}_y = 1$
$\underline{e}_x \cdot \underline{e}_y = 0$

$$= \cos(\theta_1 - \theta_2) \quad \text{(trig identity)}$$

$$\boxed{\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta_1 - \theta_2) + z_1 z_2}$$

This is correct!

* Moral of the Story: when in doubt, use cartesian coordinates For position vectors, be very careful! They are not the same if they are at different points in space.