A note on definite integrals, indefinite integrals, and BVPs.

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I. Basics:

A definite integral is given as

$$I = \int_a^b f(x)\, dx \qquad \text{for } f(x) \in [a,b]$$

The fundamental theorem of calculus states that

$$F(x) = \int_a^x f(t)\, dt$$

is the antiderivative of f, i.e. $F'(x) = f(x)$

for every $x \in (a,b)$.

When we evaluate a definite integral, we get

$$\int_a^b f(t)\, dt = F(b) - F(a) = \underbrace{F(x)\Big|_a^b = \Big[F(x)\Big]_a^b}_{\text{short-hand}}$$

using the F.T. of C.

An indefinite integral is another name for the antiderivative of a function

$$\int f(x)\, dx$$

In general, there are a family of antiderivatives for $f(x)$, that all differ by a constant.

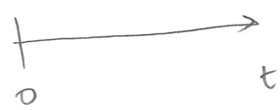
Therefore, if F is any antiderivative of f, then

$$\int f(x)\,dx = F(x) + C$$

II. Differential Equations

* Our question is really related to differential equations, and boundary value problems in particular.

A. Initial value problems



Suppose we have a linear, 1st order IVP

$$\frac{dy}{dt} = -ay + b \qquad y(0) = y_0$$

$$\downarrow$$

$$\frac{dy}{dt} \cdot \frac{1}{y - b/a} = -a$$

(1) standard way : indefinite integral

$$\int \underbrace{\frac{dy}{dt}\left(\frac{1}{y-b/a}\right) dt}_{\text{by the chain rule : } dy = \frac{dy}{dt}dt} = \int -a\,dt$$

$$\int \frac{1}{y-b/a}\,dy = \int -a\,dt$$

$$\ln(y - b/a) + C_1 = -at + C_2$$

Both arbitrary constants

$$\ln(y - b/a) = -at + C_3$$

combine for 3rd arbitrary constant

$$y - b/a = e^{(-at + C_3)}$$

$$= e^{C_3} e^{-at} = C_4 e^{-at}$$

$$y = b/a + C_4 e^{-at}$$

Now apply initial condition : $y(0) = y_0$

$$y(0) = b/a + C_4 = y_0 \Rightarrow C_4 = (y_0 - b/a)$$

$$\boxed{y = b/a + (y_0 - b/a)e^{-at}}$$

(2) Alternate way : definite integrals

$$\int_{y_0}^{y} \frac{1}{y' - b/a}\, dy' = \int_0^t -a\, dt'$$

Lower bound is at initial value. This is a valid way of taking an anti-derivative on the domain $(0, t)$

initial condition must match on both sides of the definite integral. This is like a "implicit" way of applying the IC.

$$\ln(y - b/a) - \ln(y_0 - b/a) = -at' \Big|_0^t$$

$$\ln\left(\frac{y - b/a}{y_0 - b/a}\right) = -at$$

$$\frac{y - b/a}{y_0 - b/a} = e^{-at} \Rightarrow \boxed{y = b/a + (y_0 - b/a)\, e^{-at}}$$

B. Boundary value problems

Suppose we have an integrable, 2nd order ODE BVP on the domain:

This is a special case! Most of the time we cannot separate & integrate a BVP. This is one reason that the "definite integral" approach is also misleading, when you solve an ODE in general, you end up w/ constants that you need to solve for w/ ICs or BCs.

$$\frac{d^2f}{dx^2} - 3f = 0 \qquad f(a) = 0$$
$$f(b) = 1$$

(1) standard way: indefinite integral

$$\frac{d^2f}{dx^2} - 3\frac{df}{dx} = 0$$

let $g = \frac{df}{dx}$

$$\frac{dg}{dx} - 3g = 0 \Rightarrow \frac{1}{g}\frac{dg}{dx} = 3 \Rightarrow \int \frac{1}{g}\frac{dg}{dx}\,dx = \int 3\,dx$$

$dg = \frac{dg}{dx}\,dx$ by chain rule.

$$\int \frac{1}{g}\,dg = \int 3\,dx$$

$$\ln g + C_1 = 3x + C_2$$

$$g = C_3 e^{3x} \Rightarrow \frac{df}{dx} = C_3 e^{3x}$$

$$\int \frac{df}{dx}\,dx = \int C_3 e^{3x}\,dx$$

$$f + C_4 = C_3 \frac{1}{3} e^{3x} + C_5$$

$$f = C_3 \frac{1}{3} e^{3x} + C_6 \quad \leftarrow \text{Combine } C_4 \text{ \& } C_5$$

* Now find constants using BC's

$$f(a) = \frac{C_3}{3} e^{3a} + C_6 = 0$$

$$f(b) = \frac{C_3}{3} e^{3b} + C_6 = 1$$

$$\frac{C_3}{3} e^{3a} - \frac{C_3}{3} e^{3b} = -1 \Rightarrow \frac{C_3}{3}\left(e^{3a} - e^{3b}\right) = -1$$

$$C_3 = \frac{-3}{e^{3a} - e^{3b}}$$

$$\frac{-1}{e^{3a} - e^{3b}} e^{3a} + C_6 = 0 \Rightarrow C_6 = \frac{e^{3a}}{e^{3a} - e^{3b}}$$

$$f = \frac{-e^{3x}}{e^{3a} - e^{3b}} + \frac{e^{3a}}{e^{3a} - e^{3b}} = \frac{e^{3a} - e^{3x}}{e^{3a} - e^{3b}}$$

$$\boxed{f = \frac{e^{3a} - e^{3x}}{e^{3a} - e^{3b}}}$$

(2) Alternate way: definite integrals

$$\frac{d^2 f}{dx^2} - 3\frac{df}{dx} = 0 \qquad \text{Let } g = \frac{df}{dx}$$

$$\frac{dg}{dx} = 3g \Rightarrow \int \frac{1}{g}\frac{dg}{dx}\, dx = \int 3\, dx$$

* problem: see next page

$$\Rightarrow \int \frac{1}{g}\, dg = \int 3\, dx$$

(*) Problem: We don't have a boundary condition for g!

g is the derivative! In a special case, we could do this.

Ex: $\frac{d^2f}{dx^2} - 3\frac{df}{dx} = 0$ $\quad \left.\begin{array}{l} f(0) = f_0 \\ f'(0) = g_0 \end{array}\right\}$ IVP, 2^{nd} order. Now, we could do this instead.

(*) This is the exception that Dr. Fletcher didn't see. It only arises in BVPs. You can't use definite integrals for BVPs w/ Dirichlet ($f(a)=0,\ f(b)=0$) or Neuman Boundary conditions ($f'(a)=0,\ f'(b)=0$).

Let's proceed w/ the mixed case, like we did in class

* Indefinite first:

$$\int \frac{1}{g}\, dg = \int 3\, dx$$ we've done this already.

$$g = k_1 \exp(3x)$$ ← k_1: unknown constant

* Now, we do have a value for $f(a)$ & $f(b)$, so let's try a definite integral here:

$$\frac{df}{dx} = k_1 \exp(3x) \Rightarrow \int \frac{df}{dx}\, dx = \int k_1 \exp(3x)\, dx$$

$$\int_{f(a)}^{f(x)} d\bar{f} = \int_a^x k_1 \exp(3\bar{x})\, d\bar{x}$$

again, these correspond.

$$f(x) - f(a) = \frac{k_1}{3} e^{3x} - \frac{k_1}{3} e^{3a}$$

↑

what is $f(a)$? $f(a) = 0$ (one of our BCs)

$$f(x) = \frac{k_1}{3}\left(e^{3x} - e^{3a}\right)$$ ← now, need other BC.

$$f(b) = \frac{k_1}{3}\left(e^{3b} - e^{3a}\right) = 1 \Rightarrow k_1 = \frac{3}{e^{3b} - e^{3a}}$$

$$f(x) = \frac{e^{3x} - e^{3a}}{e^{3b} - e^{3a}} \times \frac{(-1)}{(-1)}$$

$$\boxed{f(x) = \frac{e^{3a} - e^{3x}}{e^{3a} - e^{3b}}}$$

Same answer (whew!)

(3) what did we do in class?

* same first step:

$$g = k_1 \exp(3x) \Rightarrow \frac{df}{dx} = k_1 \exp(3x)$$

* now, we get messy & mix a definite integral on one side and an indefinite on the other!

$$\int df = \int_a^x k_1 \exp(3x)\, dx$$

↑ indefinite ↑ definite

* This is pretty weird. I think it is not really a clean/safe thing to do. But, it looks like it works out:

$$f + k_2 = \int_a^x k_1 \exp(3x)\, dx$$

$\uparrow$ constant on this side

$\uparrow$ nothing over here.

* The constant on the LHS should end up being $[-f(a)]$

let's work it out to check this

$$f + k_2 = \frac{k_1}{3}\left[e^{3x} - e^{3a}\right]$$

Eval BCs: $f(a) = \frac{k_1}{3}\left[e^{3a} - e^{3a}\right] - k_2 \Rightarrow k_2 = -f(a)$

(as we expected)

$k_2 = 0$

$$k_1 = \frac{3}{e^{3b} - e^{3a}}$$ $\leftarrow$ from $f(b)$ just like in (2).

* putting it all together:

$$\boxed{f = \frac{e^{3x} - e^{3a}}{e^{3b} - e^{3a}} = \frac{e^{3a} - e^{3x}}{e^{3a} - e^{3b}}}$$

* why did BSL do a definite integral? (in Ex. 4.1-1 on p. 116 Eq. 4.1-13)

Because we wanted to get the error function.

It was the easiest way.

C. Take Away lessons

* Generally, when dealing with ODEs, it always works if you use an indefinite integral. Definite integrals only work in some cases (IVPs). Also, many ODEs cannot be integrated directly.

* If you use a definite integral, match the integrand limits on both sides. Don't do like we did in class. It probably works, but it is confusing.