Lecture 8: Conservation of Energy

I. Bulk Conservation of Energy

A. Energy Equation

* Last time we derived a general property balance. Let's use this and our constitutive laws (fluxes) to get a transport equation for energy.

$$\frac{\partial b}{\partial t} + \underline{\nabla} \cdot (b\underline{v}) = -\underline{\nabla} \cdot \underline{f} + B_v \qquad (1)$$

or

$$\rho \frac{D\hat{B}}{Dt} = -\underline{\nabla} \cdot f + B_v \qquad (2)$$

b: property/volume

$\underline{f}$: diffusive flux

B_v: generation of B

$\hat{B}$: b/ρ, "specific" property

* To find the energy balance, we need to find b, $\underline{f}$, and B_v.

$$B = E = \underbrace{\frac{1}{2} m v^2}_{\text{kinetic energy}} + \underbrace{mgh}_{\text{potential energy}} + \underbrace{m\hat{u}}_{\text{internal energy}}$$

$$b = \frac{1}{2}\rho v^2 + \rho g h + \rho \hat{u}$$

$$\hat{B} = \frac{1}{2} v^2 + gh + \hat{u}$$

$$\underline{f} = \underline{q} = -k \underline{\nabla} T \quad \leftarrow \text{constitutive law}$$

$B_V = \dot{W}$ rate of work per volume

- Bird calls this H_V, rate of enthalpy generated per volume, we'll see why in a second.
- In ch. 10, we see this is: $\nabla \cdot (\underline{\sigma} \cdot \underline{v})$ ↑ total stress

like "PV work"

* Substitute into (2):

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}v^2 + gh + \hat{u}\right) = -\underline{\nabla} \cdot \underbrace{(-k \underline{\nabla} T)}_{\nabla \cdot \nabla = \nabla^2,\ k:\text{const}} + \dot{W}$$

$$\boxed{\rho \frac{D}{Dt}\left(\tfrac{1}{2}v^2 + gh + \hat{u}\right) = k\nabla^2 T + \dot{W}} \quad (3)$$

$$\underbrace{\text{"}\frac{du}{dt}\text{"}}_{} = \underbrace{\dot{Q}}_{} + \dot{W}$$

- 1st law of thermodynamics!

B. Heat Equation in terms of T

* We would like the LHS in terms of temperature. We want a PDE for $T(\underline{r}, t)$. We will solve for $\underline{v}$ ($\underline{\sigma}$) in fluid mechanics. Internal energy ($\hat{u} \leftrightarrow \hat{C}_v dT$) is the unique information here.

* TWO ways to do this:

(1) Write a separate balance for mechanical energy & subtract it from Eq. 3. Fewer assumptions, but longer. See §10.6

(2) Make a few assumptions. Shorter.

↖ Let's do #2.

* Assumption 1: $\hat{u} \gg \frac{1}{2}v^2$ & $\hat{u} \gg gh$

- This is a very good approximation.

- Example: Saturated H_2O (steam tables in Smith, van Ness, & Abbott. p. 716

$$\hat{u}_{l\to g} = 84 \frac{kJ}{kg} = 8.4 \times 10^4 \frac{m^2}{s^2} \quad (20^\circ C, \text{sat})$$

how fast? $\frac{1}{2}v^2 = \hat{u} \rightarrow v = \sqrt{2\hat{u}}$

$v = 410\ m/s$

(reference; speed of sound = $343\ m/s$)

how high? $gh = \hat{u} \rightarrow h = \hat{u}/g$

$h \approx 8500\ m$

- Assuming $\hat{u}$ dominates:

$$\rho \frac{D\hat{u}}{Dt} = k\nabla^2 T + \dot{w}$$

* Assumption 2: ρ = constant (incompressible)

- We want to get rid of $\hat{u}$ and turn it into T.
- Constant density is a pretty good assumption for liquids and for gases with $Ma \ll 1$. We'll talk more when we do momentum.

- Let's do it:

$$\frac{D\hat{u}}{Dt} = \frac{D}{Dt}\left(\hat{H} - P/\rho\right)$$

← thermo: $\hat{H} = \hat{u} + P\hat{V}$
$= \hat{u} + P/\rho$

$$= \frac{D\hat{H}}{Dt} - \frac{1}{\rho}\frac{DP}{Dt}$$

$$d\hat{H} = \hat{C}_p\, dT + \left[\frac{1}{\rho} + \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P\right] dP$$

↑ also from thermo. Eq. 10.6-16 in Deen (p. 425)

$$\frac{D\hat{u}}{Dt} = \hat{C}_p \frac{DT}{Dt} + \left[\frac{1}{\rho} + \frac{T}{\rho}\underbrace{\left(\frac{\partial \rho}{\partial T}\right)_p}_{0 \text{ if } \rho \text{ is constant}}\right]\frac{DP}{Dt} - \frac{1}{\rho}\frac{DP}{Dt}$$

$$= \hat{C}_p \frac{DT}{Dt} + \cancel{\frac{1}{\rho}\frac{DP}{Dt}} - \cancel{\frac{1}{\rho}\frac{DP}{Dt}}$$

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \dot{w}}$$

* Comments:

- Assumes constant k, ρ, and small mechanical and potential energy
- Includes convection: $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T$
- Can re-write with diffusivity

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{transient/ accumulation of thermal energy}} + \underbrace{\underline{v} \cdot \nabla T}_{\text{convection of thermal energy in/out}} = \underbrace{\alpha \nabla^2 T}_{\text{diffusion of thermal energy in/out}} + \underbrace{\frac{\dot{w}}{\rho \hat{C}_p}}_{\text{generation}}$$

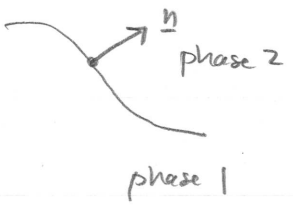
II. Heat Transfer at Interfaces

* The partial differential equation we found above will require boundary conditions. Those BC's are subject to the balance equation at interfaces we talked about last time.

$$\left[\left(\underline{f} + b(\underline{v} - \underline{v}_I)\right)_2 - \left(f + b(\underline{v} - \underline{v}_I)\right)_1\right] \cdot n_I = B_s$$

- In most cases of practical interest there is no relative flow across the interface: $\underline{v} - \underline{v}_I = 0$

- $\underline{f} = \underline{q}$, like before
- $B_S = H_S$, enthalpy generated at the interface (e.g. surface rxn)

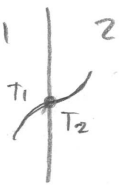


$$\underline{q}_2 \cdot \underline{n}_I - \underline{q}_1 \cdot \underline{n}_I = H_S$$

- $\underline{q} \cdot \underline{n}$: normal component of flux.
- Dean uses $q_n = \underline{q} \cdot \underline{n}$, $\frac{\partial T}{\partial n} = n \cdot \nabla T$

* let's examine some practical cases of boundary conditions we will encounter:

- case 1: Thermal Equilibrium



- Very small distances dissipate gradients very quickly. Therefore, interfaces in intimate contact have $\nabla T = 0$.

$$\underline{q}_2 \cdot \underline{n} = \underline{n} \cdot (k \nabla T)_2 = 0 \quad , T_2 = \text{const}$$

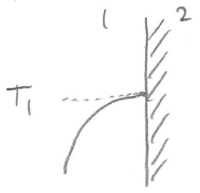
$$\underline{q}_1 \cdot \underline{n} = \underline{n} \cdot (k \nabla T)_1 = 0 \quad , T_1 = \text{const}$$

$$H_S = 0$$

$$\Rightarrow \quad q_1 \cdot \underline{n} = q_2 \cdot n \Rightarrow \boxed{T_1 = T_2}$$

- Also known as "Dirichlet" conditions in PDE literature
- Experimentally: we know the surface temperature because we have a good conductor as the boundary.

- case 2: Known Flux



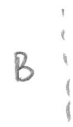
- We don't know the surface temperature, but we know the flux.

- Insulated : flux is zero

$$q_1 \cdot n = 0 \rightarrow -k \frac{\partial T}{\partial n} = 0$$

$$\boxed{\frac{\partial T}{\partial n} = 0}$$

- symmetry : flux is also zero.



$$\underline{q} \cdot \underline{n} = -\underline{q} \cdot \underline{n}$$

$$\Rightarrow 2\underline{q} \cdot \underline{n} = 0$$

$$\Rightarrow -2k \frac{\partial T}{\partial n} = 0$$

$$\boxed{\frac{\partial T}{\partial n} = 0}$$

"Neuman" in PDE literature

- heat generation, flux is known (e.g. chem rxn)

$$q_1 \cdot \underline{n} = H_s$$

$$\boxed{\frac{\partial T}{\partial n} = \frac{H_s}{-k}}$$

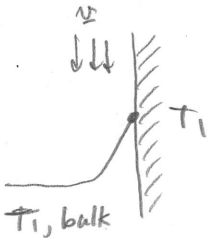
- Case 3: Flux is a function of T

- convection.

$$(\underline{q} \cdot \underline{n})_2 = (q \cdot n)_1 = h(T_1 - T_{1,bulk})$$

↑ we come up w/ correlations for this

↑ we can measure this



"Robin" in PDE literature

$$\boxed{\frac{\partial T}{\partial n} = \frac{-h}{k}(T_1 - T_{1,bulk})}$$

- nothing "fundamental" about "Newton's law of cooling": this is a definition of h, heat transfer coefficient.

- Radiation

$$(q \cdot \underline{n}) = \sigma_{SB} F_{13} (T_1^4 - T_3^4)$$

↑ Stefan-Boltzmann constant

↑ view factor

← T_3: surface radiating to / from.