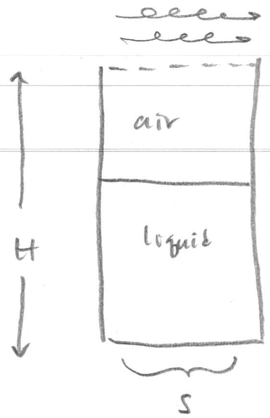
Lecture 12 - Supplement: Full Solution of liquid column evaporation



$z = h(t)$, $h(0) = 0$

$z = 0$ (gas/liq interface)

Species A: air

Species B: organic

x_A in equilibrium & stagnant.

Boundary conditions

$x_B(z=0,t) = x_0$

(from vapor pressure @ liquid surface)

$x_B(z=h(t),t) = 0$

(fast moving air keeps it 0).

* Balance Eq:

$$\frac{\partial b}{\partial t} = -\underline{\nabla} \cdot \underline{F} + R_{v,i}$$

$b = c_B$

$\underline{F} = \underline{N}_B$

$R_{v,i} = 0$

$$\frac{\partial c_B}{\partial t} = -\underline{\nabla} \cdot \underline{N}_B$$

- if $t \sim t_p$ then:

$$\tilde{t} = t/t_p \,, \quad \tilde{c}_B = c_B / c x_0$$

let $c_{B,G} = c x_0$

$$\tilde{N}_B = N_B \Big/ D_{AB} \cdot \frac{c_{B,G}}{H} \qquad \tilde{\nabla} = \nabla / H$$

$$\frac{c_{B,G}}{t_p} \frac{\partial \tilde{c}_B}{\partial \tilde{t}} = -\frac{1}{H} \cdot D_{AB} \cdot \frac{c_{B,G}}{H} \; \underline{\tilde{\nabla}} \cdot \underline{\tilde{N}}_B$$

$\frac{D_{AB}}{H^2} = \frac{1}{t_d}$

(diffusion time)

$$\frac{t_d}{t_p} \frac{\partial \tilde{c}_B}{\partial \tilde{t}} = \underline{\tilde{\nabla}} \cdot \underline{\tilde{N}}_B$$

if $t_p \gg t_d$ then we can assume pseudo steady

$$0 \simeq \underline{\tilde{\nabla}} \cdot \underline{\tilde{N}}_B$$

* Now, lets finish simplifying (back to dimensional units):

$$0 = \underline{\nabla} \cdot \underline{N}_B = \frac{\partial N_{B,x}}{\partial x} + \frac{\partial N_{B,y}}{\partial y} + \frac{\partial N_{B,z}}{\partial z}$$

• x & y are irrelevant. Either due to symetry or to different scales.

$$0 = \frac{\partial N_{B,z}}{\partial z}$$

* There is a trick to solving this problem.

(1) $N_{B,z} = c_B v_z^{(M)} + J_B^{(M)}$ ← molar units relative to $v_z^{(M)}$

$$= c_B v_z^{(M)} - C D_{AB} \frac{\partial x_B}{\partial z}$$

not zero! A is stagnant, but B is flowing.

(*) See supplemental Notes about velocities & diffusion! (Lec 10)

$N_{Az} + N_{Bz} = C\, v_z^{(M)}$ (total flux)

zero.

$$v_z^{(M)} = \frac{N_{Bz}}{c}$$

• plug back into (1)

$$N_{B,z} = c_B \cdot \frac{N_{B,z}}{c} - C D_{AB} \frac{\partial x_B}{\partial z} \qquad x_B = \frac{c_B}{c}$$

$$N_{Bz} = x_B N_{Bz} - C D_{AB} \frac{\partial x_B}{\partial z}$$

$$N_{Bz} = \frac{-C D_{AB}}{1-x_B} \frac{\partial x_B}{\partial z}$$

* Now, we can solve the ODE.

$$\frac{dN_{Bz}}{dz} = 0$$ ← easy to integrate

$$N_{B,z} = c_1$$

$$\frac{-C D_{AB}}{1-x_B} \frac{dx_B}{dz} = c_1$$ ← separable, integrate

$$\ln(x_B - 1) = \frac{c_1 z}{CDAB} + c_2$$

• substitite: $c_1 = \ln k_1$, $c_2 = \ln k_2$ } makes it easier to solve BCs.

$$\ln(x_B - 1) = \frac{z}{CDAB} \ln k_1 + \ln k_2$$

$$x_B - 1 = k_1^{z/CDAB} \cdot k_2$$

* Apply BC's:

$$x_B(z=0) = x_0 = \cancel{k_1}^{0} k_2 + 1$$

$$k_2 = x_0 - 1$$

$$x_B(z=h) = 0 = k_1^{h/CDAB} (x_0 - 1) + 1$$

$$\frac{-1}{x_0 - 1} = k_1^{h/CDAB}$$

$$k_1 = \left(\frac{-1}{x_0 - 1}\right)^{\frac{CDAB}{h}}$$

• put together:

$$x_B - 1 = \left(\frac{-1}{x_0 - 1}\right)^{\frac{z}{CDAB}\frac{CDAB}{h}} (x_0 - 1)$$

$$\frac{x_B - 1}{x_0 - 1} = \left(\frac{-1}{x_0 - 1}\right)^{\frac{z}{h}}$$

$$\boxed{\frac{1 - x_B}{1 - x_0} = \left(\frac{1}{1 - x_0}\right)^{z/h}} \leftarrow h = h(t)$$

* Now, Solve for N_{Bz}. There are two ways to do this:

* <u>Way 1</u>: Take derivative directly

$$\frac{1-x_B(z)}{1-x_0} = \left(\frac{1}{1-x_0}\right)^{z/h(t)}$$

$$N_{B,z} = -\frac{C D_{AB}}{1-x_B}\frac{dx_B}{dz}$$

$$\frac{dx_B}{dz} = -\frac{d}{dz}\left[(1-x_0)\left(\frac{1}{1-x_0}\right)^{z/h}\right] = -(1-x_0)\frac{1}{h}\left(\frac{1}{1-x_0}\right)^{z/h}\ln\left(\frac{1}{1-x_0}\right)$$

$$N_{B,z} = \frac{+C D_{AB}}{\cancel{(1-x_0)}\cancel{\left(\frac{1}{1-x_0}\right)^{z/h}}} \cdot +\cancel{(1-x_0)}\cdot\cancel{\left(\frac{1}{1-x_0}\right)^{z/h}} \cdot \frac{1}{h}\ln\left(\frac{1}{1-x_0}\right)$$

$$= \frac{C D_{AB}}{h}\ln\left(\frac{1}{1-x_0}\right) = -\frac{C D_{AB}}{h}\ln(1-x_0)$$

* <u>Way 2</u>: Implicit integration

$$N_{B,z} = -\frac{C D_{AB}}{1-x_0}\frac{dx_B}{dz}$$

$$\int_0^h \underset{\substack{\uparrow \\ \text{constant}}}{N_{B,z}}\, dz = \int_{x_0}^{0} -\frac{C D_{AB}}{1-x_B}\, dx_B$$

↑ from BC's (limits of integration)

$$N_{B,z}\cdot h = +C D_{AB}\ln(1-x_B)\Big|_{x_0}^{0}$$

$$\boxed{N_{Bz} = -\frac{C D_{AB}}{h}\ln(1-x_0)}$$ (same as way 1)

* Now, solve for $h(t)$ using expression from class:

$$\frac{dh}{dt} = \frac{N_{Bz}}{C_{B,L}} = -\frac{C\,D_{AB}}{h\,C_{B,L}} \ln(1-x_0)$$

- Let $C_{B,G} = C x_0$

$$\frac{dh}{dt} = -\frac{1}{h} \underbrace{\frac{C_{B,G}\,D_{AB}}{C_{B,L}} \frac{\ln(1-x_0)}{x_0}}_{\text{Constants.}} \qquad \underbrace{h(0) = 0}_{\text{Initial Condition}}$$

- Separable, 1st order ODE

$$\int h\,dh = -\int \frac{C_{B,G}\,D_{AB}}{C_{B,L}} \frac{\ln(1-x_0)}{x_0}\,dt$$

$$\frac{1}{2}h^2 = -\frac{C_{B,G}}{C_{B,L}} D_{AB} \frac{\ln(1-x_0)}{x_0}\, t$$

$$\boxed{h^2 = -2D_{AB}\frac{C_{B,G}}{C_{B,L}} \frac{\ln(1-x_0)}{x_0}\, t}$$

- why negative? Because $1-x_0 < 1$, so $\ln(1-x_0) < 0$. This means h^2 is positive

- Also, note that when $x_0 \ll 1$, then

$\ln(1-x_0) \approx -x_0$ and $-\frac{\ln(1-x_0)}{x_0} \approx 1$

$$h \approx \left[2D_{AB}\frac{C_{B,G}}{C_{B,L}}\, t\right]^{1/2} \text{ for } x_0 \ll 1.$$

* See python plot for full solution.