Lecture 12 - Perturbation Analysis

I. Asymptotic Expansions

* In the similarity method, we reduced a PDE to an ODE using reasoning based on scaling concepts.

* Today we are going to talk about another method based on scaling concepts: perturbation analysis.

* Unlike similarity, perturbation analysis does not reduce a PDE to an ODE. Instead, it allows us to turn an unsolvable (usually non-linear) ODE or PDE into one that we can solve.

* Perturbation methods will be critical for our understanding of boundary layers later in the course.

* The key idea for solving an ODE or PDE using perturbation methods is to use an asymptotic series.

* Asymptotic series:

finite sum

$$f(x) = \sum_{n=0}^{N} f_n(x)\,\varepsilon^n$$

"gauge function", usually a power series in a small parameter ε.

function we want to re-write

"coefficients" of the series

$$= f_0(x) + f_1(x)\varepsilon + f_2(x)\varepsilon^2 + \dots + f_N(x)\varepsilon^N$$

* To help us understand an asymptotic series, we will compare and contrast it with a Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x-x_0)^n \leftarrow \text{also a power series}$$

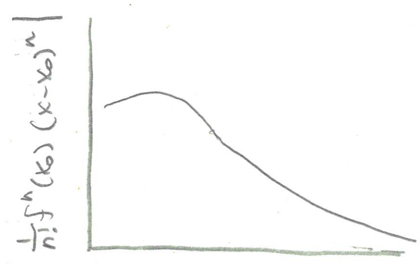
$\sum_{n=0}^{\infty}$: infinite sum

$\frac{f^n(x_0)}{n!}$: Taylor coefficients

$(x-x_0)^n$: centered at x_0

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \ldots$$

* A Taylor series is an infinite series. We use derivatives at a point, x_0, to describe the function inside some radius of convergence, R.



- The terms need to get smaller & smaller as $n \to \infty$ to converge.

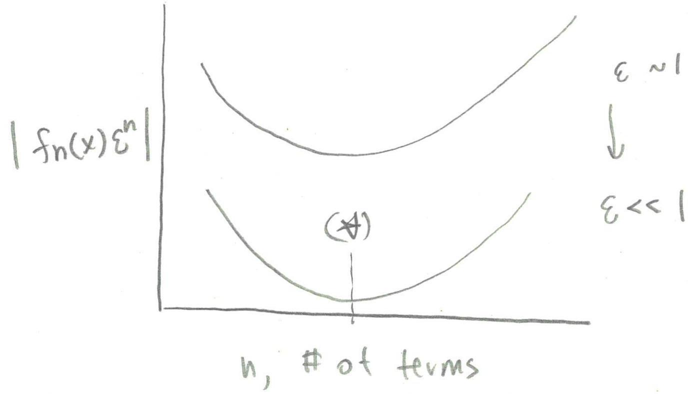
* In an asymptotic series, we are interested in a different type of convergence. We are interested in asymptotic convergence, i.e. a series that converges as $\varepsilon \to 0$.

Formally, a series that obeys,

$$\lim_{\varepsilon \to 0} \frac{f(x) - \sum_{n=0}^{N} f_n(x)\varepsilon^n}{\varepsilon^N} = 0$$

is said to converge.

* This convergence is really useful. We can use a series that diverges as $N \to \infty$, but as long as ε is small enough, it will represent the function.



(*) Truncate series here!

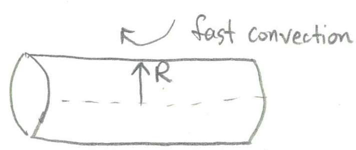
* Note: An asymptotic series does not have to diverge. It just can.

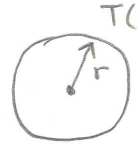
* In summary, a Taylor series is an infinite series that converges for many terms, but an asymptotic series is a finite series that converges for small ε. Consequently, this is often called "perturbation analysis", because we are using a small perturbation, ε.

II. Regular Perturbation Problems

* Like many of these methods, the best way to understand it is to do an example, and then we will generalize from there.

Example: Heated wire with temperature - dependent conductivity





$T(r=R) = T_\infty$

$\left.\dfrac{dT}{dr}\right|_{r=0} = 0$

(Ex 4.3-1 in Deen c.f. Ex 3.2-5 too)

$k(T) = k_\infty \left[1 + a(T - T_\infty) \right]$

$H_v(T) = H_\infty \left[1 + a(T - T_\infty) \right]$

* we solved this problem earlier with k, H_v constant, but with finite convection.

* i.e. $Bi = \dfrac{hR}{k_\infty} >> 1 \Rightarrow T(r=R) = T_\infty$

* Energy Balance:

$$\rho \hat{C}_p \frac{DT}{Dt} = \underline{\nabla} \cdot \left(k(T) \nabla T \right) + H_v(T) \quad , \text{ constant } \rho, \hat{C}_p$$

- steady, no convection, axisymmetric ($\frac{\partial T}{\partial \theta} = 0$), long wire / no end effects ($\frac{\partial T}{\partial z} = 0$), cylindrical coords.

$$0 = \frac{1}{r}\frac{d}{dr}\left(k(T)\, r \frac{dT}{dr} \right) + H_v(T)$$

* OOM Analysis & Non-dimensionalize:

(We need to do this for our terms to be $\mathcal{O}(1)$)

$r \sim R$, $T - T_\infty \sim ?$ $\longrightarrow$ use balance $\frac{d^2T}{dr^2} \sim \frac{T_\infty - T_c}{R^2} \sim \frac{H_\infty}{k_\infty}$

define : $\eta = r/R$, $\theta = \frac{T - T_\infty}{H_\infty R^2 / k_\infty}$ plug into balance

$$\frac{1}{\eta R} \cdot \frac{1}{R}\frac{d}{d\eta}\left(k(T)\, \cancel{R}\, \eta \cdot \frac{H_\infty R^2}{k_\infty} \frac{1}{\cancel{R}} \frac{d\theta}{d\eta} \right) + H_v(T) = 0$$

$$\frac{1}{R^2}\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\, k(T) \frac{H_\infty R^2}{k_\infty}\frac{d\theta}{d\eta} \right) + H_v(T) = 0 \qquad \times R^2/H_\infty$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta \frac{k(T)}{k_\infty}\frac{d\theta}{d\eta} \right) + \frac{H_v(T)}{H_\infty} = 0$$

$$\frac{k(T)}{k_\infty} = 1 + a(T - T_\infty) = 1 + a\frac{H_\infty R^2}{k_\infty}\theta = 1 + \varepsilon\theta$$

$$\frac{H_v(T)}{H_\infty} = 1 + a(T - T_\infty) = 1 + a\frac{H_\infty R^2}{k_\infty}\theta = 1 + \varepsilon\theta$$

let $\boxed{\varepsilon = \frac{a H_\infty R^2}{k_\infty}}$ (a has units of T^{-1})

new dimensionless group.

strength of k, H_v dependence on Temp.

$$\boxed{\frac{1}{\eta}\frac{d}{d\eta}\left[\eta(1+\varepsilon\theta)\frac{d\theta}{d\eta} \right] + (1+\varepsilon\theta) = 0}$$

B.C.'s : $\boxed{\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = 0}$, $(T - T_\infty)|_{r=R} = 0$

$\downarrow$

$\boxed{\theta(\eta=1) = 0}$

* Now, we have a math problem, a non-linear ODE, that we can't solve. It is 2^{nd} order, non-separable.

But if we let $\varepsilon = 0$, we could solve it.

$\hookrightarrow \frac{1}{\eta}\frac{d}{d\eta}\left(\eta \frac{d\theta}{d\eta}\right) = -1$ (we did this in our example before)

* In asymptotic analysis, we use this intuition, that the problem could be solved if ε is small.

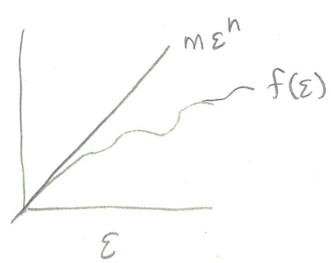
Formally, we introduce an asymptotic expansion:

$$\theta(\eta) = \underbrace{\theta_0(\eta)}_{\substack{\text{solution}\\ \text{when } \varepsilon=0}} + \underbrace{\theta_1(\eta)\,\varepsilon}_{\substack{\text{solution}\\ \text{with only } \varepsilon}} + \theta_2(\eta)\,\varepsilon^2 + \underbrace{O(\varepsilon^3)}_{\text{what is this } O(\,)\text{?}}$$

Aside: Big-O notation.

When I write $f(\varepsilon) = O(\varepsilon^n)$, I mean

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon^n} \le m, \quad \text{where } m \text{ is a constant}$$



- $m\varepsilon^n$ bounds the convergence of $f(\varepsilon)$ as $\varepsilon \to 0$.
- terms in $f(\varepsilon)$ are "smaller" than ε^n

* we do this noting that if ε is small enough, then

$$\varepsilon^3 << \varepsilon^2 << \varepsilon^1 << \varepsilon^0 = 1$$

Then, maybe only a few terms $\theta_0, \theta_1, \theta_2$ might give us a good approximation to $\theta(\eta)$.

* So, how do I solve the problem? I substitute the expansion into the ODE & boundary conditions:

balance

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta(1+\varepsilon\{\theta_0+\varepsilon\theta_1+O(\varepsilon^2)\}\frac{d}{d\eta}\{\theta_0+\varepsilon\theta_1+O(\varepsilon^2)\}\right] =$$

$$-1-\varepsilon\{\theta_0+\varepsilon\theta_1+O(\varepsilon^2)\}$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta(1+\varepsilon\theta_0+O(\varepsilon^2))\left(\frac{d\theta_0}{d\eta}+\varepsilon\frac{d\theta_1}{d\eta}+O(\varepsilon^2)\right)\right] = -1-\varepsilon\theta_0-O(\varepsilon^2)$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta\left(\frac{d\theta_0}{d\eta}+\varepsilon\theta_0\frac{d\theta_0}{d\eta}+\varepsilon\frac{d\theta_1}{d\eta}+O(\varepsilon^2)\right)\right] = -1-\varepsilon\theta_0-O(\varepsilon^2)$$

* note: whenever I have an $\varepsilon \times \varepsilon$, I can lump it into $O(\varepsilon^2)$ if I want.

B.C's

$$\theta(\eta=1) = \theta_0+\varepsilon\theta_1+O(\varepsilon^2) = 0$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = \left.\frac{d\theta_0}{d\eta}\right|_{\eta=0} + \varepsilon\left.\frac{d\theta_1}{d\eta}\right|_{\eta=0} + O(\varepsilon^2) = 0$$

* Now, ε is arbitrary, so terms of matching order on both sides of equations must be equal.

$O(\varepsilon^0)$: $\quad \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta_0}{d\eta}\right) = -1 \qquad \theta_0(1)=0, \left.\frac{d\theta}{d\eta}\right|_{\eta=0}=0$

$O(\varepsilon^1)$: $$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\,\theta_0\frac{d\theta_0}{d\eta}\right) + \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta_1}{d\eta}\right) = -\theta_0$$

$$\theta_1(1) = 0 \qquad \left.\frac{d\theta_1}{d\eta}\right|_{\eta=0} = 0$$

* we now have two problems, instead of one. The problem at $O(\varepsilon^0)$ is the easy one from before! The problem at $O(\varepsilon^1)$ looks harder, but θ_0 will already be known if we solve the $O(\varepsilon^0)$ problem first. we can do this!

* Solving the $O(\varepsilon^0)$ problem:

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta_0}{d\eta}\right) = -1 \xrightarrow{\text{integrate}} \eta\frac{d\theta_0}{d\eta} = -\frac{\eta^2}{2} + c_1$$

$$\Rightarrow \frac{d\theta_0}{d\eta} = -\frac{\eta}{2} + \frac{c_1}{\eta} \xrightarrow{\text{integrate}} \theta_0 = -\frac{\eta^2}{4} + c_1 \ln\eta + c_2$$

$\Rightarrow$ (apply BCs) $$\left.\frac{d\theta_0}{d\eta}\right|_{\eta=0} = 0 \Rightarrow c_1 = 0$$

$$\theta_0(1) = -\frac{1}{4} + c_2 = 0 \Rightarrow c_2 = 1/4$$

$$\boxed{\theta_0 = \frac{1}{4}(1-\eta^2)}$$

* Solving the $O(\varepsilon^1)$ problem:

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta_1}{d\eta}\right) = -\theta_0 - \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\,\theta_0\frac{d\theta_0}{d\eta}\right)$$

plug in θ_0 & $\frac{d\theta_0}{d\eta}$

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta\,\theta_0\frac{d\theta_0}{d\eta}\right] = \frac{1}{\eta}\frac{d}{d\eta}\left[\eta\left(\frac{1}{4}(1-\eta^2)\right)\cdot\frac{-\eta}{2}\right]$$

$$= \frac{1}{8\eta}\frac{d}{d\eta}\left[\eta^2(\eta^2-1)\right]$$

$$= \frac{1}{8\eta}\frac{d}{d\eta}\left[\eta^4 - \eta^2\right] = \frac{1}{8\eta}\left(4\eta^3 - 2\eta\right)$$

$$= \frac{\eta^2}{2} - \frac{1}{4}$$

- put with restof RHS :

$$-\theta_0 - \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\,\theta_0\frac{d\theta_0}{d\eta}\right) = -\frac{1}{4}(1-\eta^2) - \left(\frac{\eta^2}{2} - {}^1/_4\right)$$

$$= -\frac{1}{4} + \frac{\eta^2}{4} - \frac{\eta^2}{2} + \frac{1}{4} = -\frac{\eta^2}{4}$$

- simplifies to:

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta_1}{d\eta}\right) = \frac{-\eta^2}{4} \Rightarrow \frac{d}{d\eta}\left(\eta\frac{d\theta_1}{d\eta}\right) = \frac{-\eta^3}{4}$$

$$\Rightarrow \eta\frac{d\theta_1}{d\eta} = -\frac{\eta^4}{16} + c_1 \Rightarrow \frac{d\theta_1}{d\eta} = -\frac{\eta^3}{16} + \frac{c_1}{\eta}$$

$$\Rightarrow \theta_1 = \frac{-\eta^4}{64} + c_1 \ln\eta + c_2$$

- Apply BC's :

$$\left.\frac{d\theta_1}{d\eta}\right|_{\eta=0} = 0 \Rightarrow c_1 = 0$$

$$\theta_1(\eta=1) = -\frac{1}{64} + c_2 = 0 \Rightarrow c_2 = {}^1/_{64}$$

$$\boxed{\theta_1 = \frac{1}{64}(1-\eta^4)}$$

* Putting them together gives us a final solution:

$$\boxed{\theta = \frac{1}{4}(1-\eta^2) + \frac{\varepsilon}{64}(1-\eta^4) + O(\varepsilon^2)}$$

* look at solution on python to compare to numerical solution.

* Final Comments:

- A lot of algebra, but keep track of the big picture. Not that complicated, but lots of details to not mess up.

  steps:
  (1) write expansion in small parameter, ε
  (2) Substitute into problem & simplify. Do BCs too.
  (3) collect like terms to isolate problem for each order
  (4) Solve problem at each order.
  (5) Collect final solution

- We can continue this procedure indefinitely to higher orders. More algebra.

- We have an approximate solution that is only true when $\varepsilon \ll 1$. This is the perturbation analysis. We have a temperature-dependent k & H_v, but their dependence is weak.

- This is more useful than you might think. Sometimes the answer is good even if $\varepsilon \approx 1$. Sometimes we say this is an "abuse" of a perturbation solution.

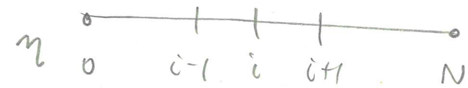
- Perturbation analysis is a formalized way to do OOM solutions. We need terms to be ~ 1 to do this.

- Perturbation analysis is extremely common. It is one of the most powerful analytical techniques to solve non-linear problems. Provides a rational, systematic way to attack them.

- P.A. is behind the phenomena of boundary layers & the pseudo-steady-state hypothesis you use in kinetics

Python Notes – Finite Difference Solution

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta(1+\varepsilon\theta)\frac{d\theta}{d\eta}\right]+1+\varepsilon\theta=0$$



$$\left.\frac{d\theta}{d\eta}\right|_i = \frac{\theta_{i+1/2}-\theta_{i-1/2}}{\Delta\eta}$$

$$\frac{d}{d\eta}\left[\eta(1+\varepsilon\theta)(\theta_{i+1/2}-\theta_{i-1/2})\cdot\frac{1}{\Delta\eta}\right] = \Big\{\left[\eta(1+\varepsilon\theta)(\theta_{i+1/2}-\theta_{i-1/2})\right]_{i+1/2} - \left[\eta(1+\varepsilon\theta)(\theta_{i+1/2}-\theta_{i-1/2})\right]_{i-1/2}\Big\} \times \frac{1}{\Delta\eta^2}$$

(*) $$\frac{1}{\eta_i\Delta\eta^2}\left[\eta_{i+1/2}(1+\varepsilon\theta_{i+1/2})(\theta_{i+1}-\theta_i)\right] - \left[\eta_{i-1/2}(1+\varepsilon\theta_{i-1/2})(\theta_i-\theta_{i-1})\right] = -1-\varepsilon\theta_i$$

BC.s

$$\theta(\eta=1)=0 \rightarrow \theta_{N-1}=0$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0}=0 \Rightarrow \frac{\theta_1-\theta_0}{\Delta\eta}=0 \Rightarrow \theta_0=\theta_1$$

Let $\eta_{i+1/2}=\frac{1}{2}(\eta_i+\eta_{i+1})$ $\quad \eta_{i-1/2}=\frac{1}{2}(\eta_i+\eta_{i-1})$ ✓ interpolation

$\theta_{i+1/2}=\frac{1}{2}(\theta_i+\theta_{i+1})$ $\quad \theta_{i-1/2}=\frac{1}{2}(\theta_i+\theta_{i-1})$

Rewrite (*):

$$\eta_{i+1/2}(1+\varepsilon\theta_{i+1/2})(\theta_{i+1}-\theta_i) - \eta_{i-1/2}(1+\varepsilon\theta_{i-1/2})(\theta_i-\theta_{i-1}) + \eta_i\Delta\eta^2(1+\varepsilon\theta_i)=0$$

* If const k:

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta}{d\eta}\right)+1=0 \Rightarrow \frac{1}{\eta_i\Delta\eta^2}\left[\eta_{i+1/2}(\theta_{i+1}-\theta_i)-\eta_{i-1/2}(\theta_i-\theta_{i-1})\right]+1=0$$