Lecture 13 - Linear PDEs and the Finite Fourier Transform

I. Intro to linear PDEs

* OK, we've done all we can with simplifications, and now we need to directly deal with PDEs.

* We will limit our analysis for now to linear PDEs. For the most part, linear PDEs are ones that we can solve.

   - what is a linear PDE? A PDE whose differential operator & initial / boundary conditions are linear. The dependent variable $f(\underline{r},t)$ is not involved in any products (f^2 or $f\frac{\partial f}{\partial x}$) nor is it inside the argument of a transcendental function ($\exp(f)$).

   - What properties does a linear PDE have?

      - It's solutions are unique. That is, if we find a solution $f(\underline{r},t)$, it is the only one.

      - It's solutions obey the principle of superposition. That is if $f_1(\underline{r},t)$ and $f_2(\underline{r},t)$ are solutions then

        $f(\underline{r},t) = c_1 f_1(\underline{r},t) + c_2 f_2(\underline{r},t)$ is also a solution.

      - These do not contradict. A unique solution to a PDE is determined by the differential operator and initial / boundary conditions. However, solutions of the differential operator only can be superposed. This allows us to decompose problems to easier ones sometimes.

* we often adopt a more compact <u>operator</u> <u>notation</u> for linear PDEs:

$$\mathcal{L} f = S(\underline{r})$$

$f(\underline{r},t)$

differential operator

$S(\underline{r})=0$, homogeneous

$S(\underline{r})\neq 0$, inhomogeneous

example:

$$\frac{\partial c}{\partial t} = D\nabla^2 c - k_1 c \quad \text{(linear reaction diffusion)}$$

$$\mathcal{L} = \left(\frac{\partial}{\partial t} - \nabla^2 - k_1\right), \quad f = c, \quad S(\underline{r}) = 0$$

- A PDE must have initial & boundary conditions to be well-posed. There are three types of BCs for a linear PDE:

  - Dirichlet (value): $f = f_s(\underline{r}_s, t)$ $\quad \underline{r}_s$: on surface.
  - Neumann (derivative): $\underline{n}\cdot\underline{\nabla} f = g_s(\underline{r}_s, t)$
  - Robin (mixed): $\underline{n}\cdot\underline{\nabla} f + h_1(\underline{r}_s,t) f = h_0(\underline{r}_s,t)$

- To have an interesting solution, something must be inhomogeneous (PDE or initial / boundary conditions). Else the solution is "trivial", $f=0$. If more than one is inhomogeneous, we often break it into two problems & solve separately (superposition!).

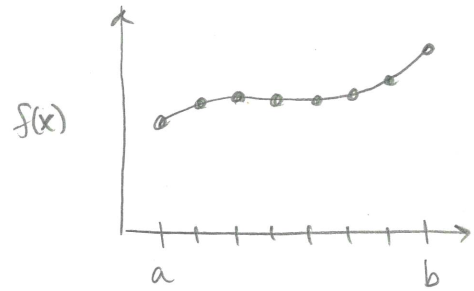
II. Function - Vector Analogy

* This operator notation looks suspiciously like the operator notation in linear algebra:

$$\mathcal{L} f = S \quad \Longleftrightarrow \quad A x = b$$

* This is not an accident. There are strong parallels between these two cases. While we could be much more vigorous, we are going to explain this with a function - vector analogy.

- In our analogy, a function is an infinite - dimensional version of a vector



$\underline{f} = [f_1, f_2, \dots, f_n]$

$i = 1, 2, \dots, n$

$x_i = a + \left(\frac{b-a}{n+1}\right) i$

$$f(x) = \lim_{n \to \infty} f_n$$

- This analogy extends further. Just like there are vector spaces, there are function spaces.

vector space: $\mathbb{R}^2$

function space: infinite dimensional ?!

$L_2(a,b)$: space of functions where $\int_a^b |f(x)|^2 dx < \infty$.

A type of Hilbert space.

- Function spaces can have a basis set that span the space.

vectors: $\underline{v} = \sum_{i=1}^{n} c_i \underline{e}_i$

component   basis vectors

functions: $f(x) = \sum_{i=1}^{\infty} c_i \psi_i(x)$

coefficient   basis function

- The most useful basis functions are orthogonal and normal. To define these properties we need an inner product.

vectors: $\underline{e}_i \cdot \underline{e}_j = 0$ orthogonal

$\underline{e}_i \cdot \underline{e}_i = 1$ normal

functions:

$$(f,g) = \int_a^b w(x)\, f(x)\, g(x)\, dx = 0, \quad \text{orthogonal}$$

$$(f,f) = \int_a^b w(x)\, f(x)^2\, dx = 1, \quad \text{normal}$$

(f,g) is a function innerproduct.
$w(x)$ is a weight function.
($w(x)=1$ for many function spaces.)

- If we have an orthogonal basis, we can easily find the coefficients for our "component notation"

$$f(x) = \sum_{i=1}^{\infty} c_i\, \psi_i(x)$$

take inner product with $\psi_j(x)$

$$(f, \psi_j) = \sum_{i=1}^{\infty} c_i\, (\psi_i, \psi_j)$$

if ψ_i are orthogonal then $(\psi_i, \psi_j) = 0$ when $i \neq j$

$$\boxed{c_i = \frac{(f, \psi_i)}{(\psi_i, \psi_i)}}$$

How does this formula change when normal? (Denom $= 1$)

- Finally, linear differential operators are analogous to the <u>matrix</u> in linear algebra:

  e.g. $\dfrac{d^2 f}{dx^2} = 0 \qquad \mathcal{L} = \dfrac{d^2}{dx^2}$

  $f(a)=0, \; f(b)=0$

- consider the finite difference for $n=5$

$i=0$	1	2	3	4	
$x=0$	x_1	x_2	x_3	b	
$f=f_0$	f_1	f_2	f_3	f_4	

$$\frac{d^2 f}{dx^2} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

for $i = 1, 2, 3$

$f_1 = f_5 = 0$

$$\frac{1}{\Delta x^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{\Delta}}_2 \cdot \underline{f} = \underline{0}$$

- $\displaystyle\lim_{n\to\infty} \underline{\underline{\Delta}}_2 = \frac{d^2}{dx^2}$ plus boundary conditions

- Note that the operator includes the BCs.
  Different BCs $\rightarrow$ Different operator.

III. Eigenvalue Problems & Fourier Series

* When a homogeneous differential equation has homogeneous boundary conditions, we often get an eigenvalue problem:

$$\mathcal{L}\psi = \lambda \psi \qquad \text{(analogous to } \underline{\underline{A}}\,\underline{x} = \lambda\, \underline{x}\text{)}$$

- In linear algebra, if the matrix satisfies certain conditions, the eigenvectors form a complete, orthogonal basis set:

$$\underline{\underline{A}}\,\underline{x}_i = \lambda_i \underline{x}_i \quad \text{for } i = 1, 2, \ldots, n$$

$$\underline{y} = \sum_{i=1}^{n} c_i \underline{x}_i$$ eigenvectors as basis set.

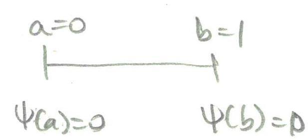
- We can do the same thing with functions:

$$\mathcal{L}\psi_i = \lambda_i \psi_i$$

$$f(x) = \sum_{i=1}^{\infty} c_i \psi_i(x)$$ eigenfunctions as a basis set.

- Example: let's find a basis set!

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad \psi(0) = 0, \; \psi(1) = 0$$



- an eigenvalue problem for $\mathcal{L} = \frac{d^2}{dx^2}$ with homogeneous Dirichlet conditions

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$ - linear, homogeneous, 2nd order ODE, constant coefficients

$$r^2 + k^2 = 0$$

$$r = \pm ik$$

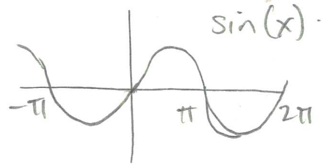
$$\psi = c_1 \sin(kx) + c_2 \cos(kx)$$

- apply BC's

$$\psi(0) = c_2 \cos(kx) = 0 \qquad c_2 = 0$$

$$\psi(1) = c_1 \sin(k) = 0$$ if $c_1 = 0$ only trivial solution!

so $c_1 \neq 0$



$\sin(x) = 0$ at $x = \pm n\pi$

$k = \pm n\pi$

$$\boxed{\psi(x) = c_1 \sin(n\pi x) \quad , \text{ for } n = 0, \pm1, \pm2, \ldots \infty}$$

- Infinite number of them!
- Are they orthogonal?

Integral table or mathematica

$$\int_0^1 c_1^2 \sin(n\pi x)\sin(m\pi x)\, dx = 0 \quad \text{if } n \neq m!$$

- what about c_1? Let's use it to normalize the set!

$$\int_0^1 c_1^2 \sin^2(n\pi x)\, dx = \frac{c_1^2}{2}$$

let $c_1^2/2 = 1 \Rightarrow c_1 = \sqrt{2}$

$$\boxed{\psi(x) = \sqrt{2}\sin(n\pi x) \quad , n = 0, \pm1, \pm2, \ldots}$$

* Now that we have a basis set to play with, let's look back at "component notation".

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x) \leftarrow \text{basis functions from above}$$

$$f(x) = \sum_{n=1}^{\infty} c_i \sqrt{2}\sin(n\pi x)$$

* This is a Fourier Series!!

we can write any function as a sum of sines!

* In fact there are many different basis functions we can use. Our "component notation" is called a "generalized Fourier Series."

* How do we get the coefficients? Use our inner product:

$$c_i = \frac{(f_i, \varphi_i)}{(\varphi_i, \varphi_i)} = \int_0^1 f(x)\sqrt{2}\sin(n\pi x)\,dx$$

we did this already:

$$q^2\int_0^1 \sin(n\pi x)\sin(m\pi x) = 1$$

$$c_n = \int_0^1 f(x)\sqrt{2}\sin(n\pi x)\,dx$$

* Together, the component notation & the coefficient form a forward and inverse Finite Fourier Transform (FFT):

$$c_n = \int_a^b f(x)\,\varphi_n(x)\,dx$$

eigenfunctions that are orthonormal.

(Forward FFT)

$$f(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

(Inverse FFT)

* In the next section, we are going to use this to solve PDEs.

How is a finite Fourier Transform related to separation of variables & Fourier Transforms?

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad \begin{aligned} &\theta(0,y) = 0 \\ &\theta(1,y) = 0 \\ &\theta(x,0) = x(1-x) \end{aligned}$$

* on an infinite domain

$$\hat{\theta}(k,y) = \int_{-\infty}^{\infty} \theta(x,y)\, e^{-ikx}\, dx$$ Fourier Transform

$$\theta(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(k,y)\, e^{ikx}\, dk$$ Inverse Fourier Transform

↳ like fourier series.

* On a finite domain

$$\hat{\theta}_n(y) = \int_0^1 \theta(x,y)\, \sqrt{2}\sin(n\pi x)\, dx$$ Finite Fourier Transform

$$\theta(x,y) = \sum_{n=1}^{\infty} \hat{\theta}_n(y)\, \sqrt{2}\sin n\pi x$$ "Inverse" Finite Fourier Transform. ⇒ Fourier Series

↳ can prove this is inverse by using orthogonality. Same thing w/ traditional Fourier Transform.

* 2 ways to solve:

Substitute Fourier Series

OR

Take Finite Fourier Transform. ← need for initial or boundary conditions

↑ (How Deen Does it)