# Lecture 15 – The Navier Stokes Equation

* Up until now in class, we have focused on heat and mass transport. We have spent several weeks thinking about mathematical techniques to solve these problems.

* Now, we turn our attention to momentum transport, i.e. fluid mechanics. We will learn something about fluids and then we will put all three together!

## I. The Cauchy Momentum Equation

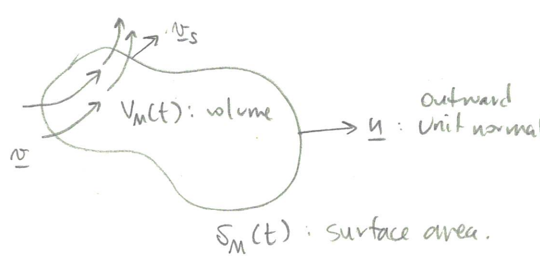
### A. A momentum balance.

* We first need to derive an equation for momentum transport. To do this we will start with a momentum balance.

* For a particle, you know momentum conservation as Newton's 2nd Law:

$$\frac{d(m\underline{v})}{dt} = \underline{F}$$

* we can re-write this for a material volume. A material volume is a control volume of arbitrary shape with constant mass.




  • constant mass implies that $\underline{v}_s = \underline{v}$.

$$\frac{d}{dt}\int_{V_M(t)} \rho \underline{v}\, dV = \underline{F}$$

* We want this in a differential form. Let's use what we learned before to manipulate this.

- Leibniz rule: $\frac{d}{dt}\int_{V(t)} b\, dV = \int_{V(t)} \frac{\partial b}{\partial t}\, dV + \int_{S(t)} b\, \underline{n}\cdot\underline{v}_s\, dS$

$$\Rightarrow \int_{V_m(t)} \frac{\partial}{\partial t}(\rho\underline{v})\, dV + \int_{S_M(t)} \rho\underline{v}\underbrace{(\underline{n}\cdot\underline{v})}_{}\, dS = \underline{F}$$

$$\rho v_i n_j v_j = n_j(\rho v_j v_i) = \underline{n}\cdot(\rho\underline{v}\,\underline{v})$$

- Divergence theorem for tensors: $\int_{S(t)} \underline{n}\cdot\underline{\underline{T}}\, dS = \int_{V(t)} \underline{\nabla}\cdot\underline{\underline{T}}\, dV$

$$\Rightarrow \int_{V_m(t)} \underbrace{\left[\frac{\partial}{\partial t}(\rho\underline{v}) + \underline{\nabla}\cdot(\rho\underline{v}\,\underline{v})\right]}_{} dV = \underline{F}$$

product rule $\rho\underline{v}, \underline{v}$

$$\underbrace{\rho\frac{\partial\underline{v}}{\partial t} + \rho\underline{v}\cdot\underline{\nabla}\underline{v}}_{\rho\frac{D\underline{v}}{Dt}} + \underbrace{\underline{v}\frac{\partial\rho}{\partial t} + \underline{v}\,\underline{\nabla}\cdot(\rho\underline{v})}_{\underline{v}\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v})\right]}$$

$\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v})\right] \to 0$, continuity

$$\Rightarrow \int_{V_m(t)} \rho\frac{D\underline{v}}{Dt}\, dV = \underline{F}$$

## B. Forces

* There are two kinds of forces that can act on a fluid:

- body forces that act everywhere in the volume (gravity, EM)
- surface forces that act on a surface (pressure, stress)

$$\underline{F} = \underline{F}_V + \underline{F}_S$$

$$\underline{F}_V = \int_{V_M(t)} \rho\underline{g}\, dV \qquad \text{gravitational}$$

$$\underline{F}_S = \int_{S_M(t)} \underline{n}\cdot\underline{\underline{\sigma}}\, dS \qquad \text{stress.}$$

* $\underline{\underline{\sigma}}$ is a stress tensor. A stress vector $\underline{s} = \underline{n} \cdot \underline{\underline{\sigma}}$ is the stress on a surface with unit normal $\underline{n}$.

The stress tensor lets us define the stress for <u>any</u> <u>direction</u>.

- The stress tensor is symmetric. (exception ferrofluids). This is a consequence of the conservation of angular momentum.

  torque's from magnetic field.

- Pressure is an isotropic, normal stress that exists even when the fluid is stationary. So, we use this to define a <u>viscous</u> <u>stress</u> <u>tensor</u>

$$\underline{\underline{\sigma}} = -P\,\underline{\underline{\delta}} + \underline{\underline{\tau}}$$

pressure is <u>in</u>. $\underline{n}$ is out.

isotropic & normal

definition of $\underline{\underline{\tau}}$.

- P: same as thermodynamic pressure when compressible i.e. $P = P(\rho, T)$
- when incompressible, P is a <u>mechanical</u> quantity only. (Lagrange multiplier)

* Let's put these together:

$$\underline{F} = \int_{V_M(t)} \rho \underline{g} \, dV + \int_{S_M(t)} \underline{n} \cdot (-P\underline{\underline{\delta}} + \underline{\underline{\tau}}) \, dV$$

divergence theorem

$$= \int_{V_M(t)} \Big[ \rho \underline{g} - \underbrace{\underline{\nabla} \cdot (P\underline{\underline{\delta}})}_{\underline{\nabla} \cdot (P\underline{\underline{\delta}}) = \underline{\nabla} P} + \underline{\nabla} \cdot \underline{\underline{\tau}} \Big] dV$$

$$= \int_{V_M(t)} \left[ \rho \underline{g} - \underline{\nabla} P + \underline{\nabla} \cdot \underline{\underline{\tau}} \right] dV$$

* Combine this with part A:

$$\int_{V_M(t)} \left[ \rho \frac{D\underline{v}}{Dt} - \rho \underline{g} + \underline{\nabla} P - \underline{\nabla} \cdot \underline{\underline{\tau}} \right] dV = 0$$

* In the limit that $V_M(t) \to 0$, or for arbitrary $V_M(t)$, the quantity in the brackets has to be 0.

$$\boxed{\rho \frac{D\underline{v}}{Dt} = \rho \underline{g} - \nabla P + \underline{\nabla} \cdot \underline{\underline{\tau}}}$$

* Comments:

  - This is the Cauchy momentum Equation.
  - A differential momentum balance. Equivalent to Newton's 2nd law.
  - We cannot solve this equation until we know how $\tau$ relates to $\underline{v}$.
  - This is a vector equation. One for $v_x$, $v_y$, & $v_z$ (for example) in Cartesian coordinates.

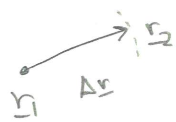
## II. Constitutive Laws for Momentum

### A. Fluid Kinematics

* To understand how stresses deform a fluid, we need to know a bit about how a fluid moves. This is called kinematics.

* Consider the Taylor series for the velocity about a point, $\underline{r}_1$ in space:

(General T.S.) $$\underline{f}(\underline{r}_2) = \underline{f}(\underline{r}_1) + \underbrace{(\underline{r}_2 - \underline{r}_1)}_{\Delta \underline{r}} \cdot \underline{\nabla} \underline{f}(\underline{r}_1) + O(|\Delta r|^2)$$



$$\underline{v}(r_2) = \underline{v}(r_1) + \Delta \underline{r} \cdot \nabla \underline{v}(r_1) + O(|\Delta r|^2)$$

focus on this term

$$\nabla \underline{v} = \underline{\underline{\Gamma}} + \underline{\underline{\Omega}}$$

$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\nabla \underline{v} + (\nabla \underline{v})^t\right]$ symmetric part

$\underline{\underline{\Omega}} = \frac{1}{2}\left[\nabla \underline{v} - (\nabla \underline{v})^t\right]$ anti-symmetric part

↳ i.e. $\underline{\underline{\Gamma}} = \underline{\underline{\Gamma}}^t$ ; $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^t$

$$\boxed{\underline{v}(r_2) = \underline{v}(r_1) + \Delta \underline{r} \cdot \underline{\underline{\Gamma}} + \Delta \underline{r} \cdot \underline{\underline{\Omega}} + O(|\Delta r|^2)}$$

translation deformation rotation

- this represents 3 ways a fluid can move : translation, deformation or rotation
- Book proves these modes are what I say. I don't have time for proof, but see §6.4 for proof.
- $\underline{\underline{\Gamma}}$: rate of strain tensor, $\underline{\underline{\Omega}}$: vorticity tensor

* Two more things we can say to help us understand consitutive laws :

- An antisymmetric tensor can only have 3 independent elements : $\begin{bmatrix} 0 & w_z & -w_y \\ -w_z & 0 & w_x \\ w_y & -w_x & 0 \end{bmatrix}$ So, it is often more convenient to re-write $\underline{\underline{\Omega}}$ using a <u>vector</u>. This vector is called the <u>vorticity</u> :

$$\underline{\underline{\Omega}} = \tfrac{1}{2}(\varepsilon \cdot \underline{w})$$

$$\underline{\underline{\Omega}} = \tfrac{1}{2}\varepsilon_{ijk} w_k$$

permutation symbol from cross product

Proof in Deen about how this implies rigid body rotation.

- A fluid element can deform in two ways: shape change and volume change. The latter is called <u>dilation</u>.

We have already seen dilation, but let's be more explicit:

$$\frac{\Delta \text{vol}}{\Delta \text{time}} = \text{area} \cdot \text{velocity}$$

$$\frac{dV}{dt} = \int_S \underline{n} \cdot \underline{v} \, dS$$

$$\frac{dV}{dt} = \int_V \underline{\nabla} \cdot \underline{v} \, dV$$

$$\frac{1}{V}\frac{dV}{dt} = \frac{1}{V}\int_V \underline{\nabla} \cdot \underline{v} \, dV = \langle \underline{\nabla} \cdot \underline{v} \rangle$$

$$\lim_{V \to 0} \frac{1}{V}\frac{dV}{dt} = \underline{\nabla} \cdot \underline{v}$$ : rate of change of volume at a single point. → dilation.

- the rate of dilation often gets a factor of $1/3$ in front of it because this is the average in 1 direction. $\underline{\nabla} \cdot \underline{v} \to$ Sum of 3 directions. $\frac{1}{3}(\underline{\nabla} \cdot \underline{v}) \to$ avg

## B. Newton's Law of Viscosity

* Now, we are ready to write down a constitutive law for a fluid. Newton's key physical insight was that <u>forces should be proportional to a rate of deformation in a fluid</u>.

e.g. solid : force ⇔ deformation

fluid : force ↔ rate of deformation

* So, $\underline{\underline{\tau}}$, the stress that exists when a fluid is moving only, should cause a rate of deformation, $\underline{\underline{\Gamma}}$.

* mathematically, we say:

$$\underline{\underline{\tau}} \propto \underline{\underline{\Gamma}}$$

- in general, $\underline{\underline{\tau}} = \underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\Gamma}}$ where $\underline{\underline{\underline{\underline{A}}}}$ is a $4^{th}$ order tensor! $\nearrow$ 81 components!

- If $\underline{\underline{\tau}}$ is symmetric (which it must be) & isotropic (which it often is), then one can show:

$$\boxed{\underline{\underline{\tau}} = \underbrace{2\mu\left[\underline{\underline{\Gamma}} - \tfrac{1}{3}(\nabla\cdot\underline{v})\,\underline{\underline{\delta}}\right]}_{\text{shape changing part}} + \underbrace{3K\left[\tfrac{1}{3}(\nabla\cdot\underline{v})\,\underline{\underline{\delta}}\right]}_{\text{volume changing part}}}$$

- Only 2 constants: $\mu$ – bulk viscosity
  $K$ – dilational viscosity

- Usually $K \ll \mu$. Also, for incompressible $\nabla\cdot\underline{v} = 0$!

* So, for an isotropic, incompressible fluid

$$\boxed{\underline{\underline{\tau}} = 2\mu\,\underline{\underline{\Gamma}}}$$

* Of course, there are many fluids that are not so simple. These have some anisotropy or solid-like character. We won't say more about those in this class :. (Rheology!)

## III. The Navier Stokes Equation

### A. Putting it all together

* Now that we know $\underline{\underline{\tau}}$, let's combine with the Cauchy momentum equation:

$$\rho \frac{D\underline{v}}{Dt} = \rho \underline{g} - \underline{\nabla} P + \underbrace{\underline{\nabla} \cdot \underline{\underline{\tau}}}_{\underline{\nabla}\cdot(2\mu \underline{\underline{\Gamma}})}$$

$$\underline{\nabla} \cdot (2\mu \underline{\underline{\Gamma}}) = 2\mu \cdot \frac{1}{2} \left[ \underline{\nabla} \cdot (\underline{\nabla}\underline{v}) + \underline{\nabla} \cdot (\underline{\nabla}\underline{v})^t \right]$$

$$= \mu \left[ \partial_i (\partial_i v_j) + \underbrace{\partial_i (\partial_j v_i)}_{\partial_j (\partial_i v_i)} \right]$$

swith order of derivative

$$= \mu \left[ \nabla^2 \underline{v} + \underline{\nabla}(\cancel{\underline{\nabla}\cdot\underline{v}}) \right]$$

0, already assumed incompressibility

$$= \mu \nabla^2 \underline{v}$$

$$\boxed{\rho \frac{D\underline{v}}{Dt} = \rho \underline{g} - \underline{\nabla} P + \mu \nabla^2 \underline{v}}$$

* <u>Comments</u>:

- This is the famous Navier-Stokes equation.
- Assumes Newtonian fluid & incompressible. Go back to CME if not.
- we often use a dynamic pressure:

$$\underline{\nabla} \mathcal{P} \equiv \underline{\nabla} P - \rho \underline{g}$$

(static pressure equation is $\underline{\nabla} P = \rho \underline{g}$ or $\underline{\nabla}\mathcal{P} = 0$)

$$\boxed{\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla}\mathcal{P} + \mu \nabla^2 \underline{v}}$$

- with the continuity equation ($\underline{\nabla}\cdot\underline{v} = 0$), this gives us 4 equations & 4 unknowns for $\underline{v}$ & $P$. Can now solve for $\underline{v}$.

- N-S. is a non-linear PDE. $2^{nd}$ order in space, $1^{st}$ order in time. No general solution. Solutions are not unique in general. Very hard PDE to solve. Entire careers dedicated to types of solutions.

- The best we can do is learn some classes of simplifications to help us start solving it. Even then we often have a PDE or perturbation problem to solve.

## B. Dimensional Analysis of Navier Stokes

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla} \mathcal{P} + \mu \nabla^2 \underline{v}$$

* Pick scales for fundamental dimensions: length, time, force/energy

$$\text{length} \downarrow L, \quad \text{time} \downarrow L/u, \quad \text{force/energy} \downarrow \Pi \cdot L^2$$

will come from BC's ← [domain size, convection scale] ? unknown pressure scale.

- Note: Deen has an extra "process" time scale, $t_p$. This isn't strictly necessary for now.

* Define re-scaled variables:

$$\tilde{t} = tu/L, \quad \tilde{\underline{v}} = \underline{v}/u, \quad \tilde{\underline{r}} = \underline{r}/L, \quad \tilde{\nabla} = \nabla L, \quad \tilde{\mathcal{P}} = \mathcal{P}/\Pi$$

$$\rho \frac{u}{L/u} \frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\frac{\Pi}{L} \tilde{\nabla} \tilde{\mathcal{P}} + \frac{\mu u}{L^2} \tilde{\nabla}^2 \tilde{\underline{v}}$$

↓ multiply by L to put in units of stress

$$\underbrace{\rho u^2 \frac{D\tilde{\underline{v}}}{D\tilde{t}}}_{\text{convective/inertial stress}} = -\Pi \tilde{\nabla} \tilde{\mathcal{P}} + \underbrace{\frac{\mu u}{L} \tilde{\nabla}^2 \tilde{\underline{v}}}_{\text{viscous stress}}$$

* There are <u>two</u> possible stress scales. Both are equally valid. Let's pick one & see what happens.

* Let $\Pi = \mu U/L$, viscous stress scale

$$\rho U^2 \frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\frac{\mu U}{L}\tilde{\underline{\nabla}}\tilde{P} + \frac{\mu U}{L}\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\frac{\rho U^2}{\mu U/L}\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\underline{\nabla}}\tilde{P} + \tilde{\nabla}^2\tilde{\underline{v}}$$

$$\rho U^2 \frac{L}{\mu U} = \frac{\rho U L}{\mu} = Re$$

$$Re\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\underline{\nabla}}\tilde{P} + \tilde{\nabla}^2\tilde{\underline{v}}$$

- This form is useful when $\mu U/L \gg \rho U^2$, i.e. when the pressure is dominated by viscous forces. When this is the case $Re \ll 1$.

- Letting $Re \to 0$ gives

$$\boxed{\begin{gathered} 0 = -\tilde{\underline{\nabla}}\tilde{P} + \tilde{\nabla}^2\tilde{\underline{v}} \\ \text{or} \\ 0 = -\underline{\nabla}P + \mu\nabla^2\underline{v} \end{gathered}}$$

This is called Stokes Equation

- It is <u>linear</u>! Good for slow flows of small things. "<u>Creeping flow</u>."

* Let $\Pi = \rho U^2$, inertial / convective scale.

$$\rho U^2 \frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\rho U^2\tilde{\underline{\nabla}}\tilde{P} + \frac{\mu U}{L}\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\frac{D\tilde{\underline{v}}}{D\tilde{t}} = -\tilde{\underline{\nabla}}\tilde{P} + \frac{\mu U/L}{\rho U^2}\tilde{\nabla}^2\tilde{\underline{v}}$$

$$\frac{D\tilde{v}}{Dt} = -\tilde{\nabla}\tilde{p} + \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{v}}$$

- This form is useful when the pressure is dominated by inertial forces. This happens when $Re \gg 1$.

- letting $Re \to 0$ gives

$$\boxed{\begin{gathered}\frac{D\tilde{v}}{D\tilde{t}} = -\tilde{\nabla}\tilde{p} \\ \text{or} \\ \rho\frac{Dv}{Dt} = -\nabla p\end{gathered}}$$

This is called Euler's Equation

- This is not linear, but we will be able to make another assumption to get there. This is called <u>inviscid flow</u>. It is good for big objects moving fast.

* Our next several lectures will talk about how to solve the Navier-Stokes, Stokes, & Euler Equations.