Lecture 16 - Unidirectional Flow

I. Why unidirectional?

* When trying to solve the Navier-Stokes equation, we have two problems:

  (1) It is non-linear, i.e. we have $\underline{v} \cdot \underline{\nabla} \underline{v}$ inside $\frac{D\underline{v}}{Dt}$.

  (2) we don't have an explicit equation for pressure, only the continuity equation.

* Flows that are parallel, i.e. <u>unidirectional</u> solve this problem.

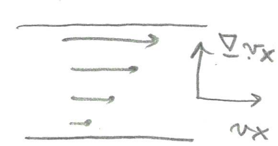
  - directions = # of non-zero velocity components : v_x, v_y, v_z
  - dimensions = # of coordinates : x, y, z
  - unidirectional means only 1 non-zero v_x, v_y, v_z.

(1) $\underline{v} \cdot \underline{\nabla} \underline{v} = 0$ for unidirectional flows

- to satisfy conservation of mass, $\underline{v}$ & $\underline{\nabla} \underline{v}$ must be in different directions, e.g.

$$v_x \neq 0, \quad v_y = v_z = 0$$

$$\underline{v} \cdot \underline{\nabla} \underline{v} = v_x \frac{\partial v_x}{\partial x}$$



$$\underline{\nabla} \cdot \underline{v} = \frac{\partial v_x}{\partial x} = 0$$

$\hookrightarrow v_x = v_x(y, z)$

$$\text{so,} \quad \underline{v} \cdot \underline{\nabla} \underline{v} = 0$$

(2) Pressure is usually greatly simplified. For example

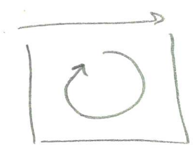
in cartesian: $\frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \Rightarrow P = P(x)$ only

when $v_x \neq 0$ & $v_y = v_z = 0$ (Navier stokes simplifies to)

* Unidirectional flows cannot help in some cases. These include:

(1) Problems where the geometry doesn't allow it.

For example, a lid driven cavity:



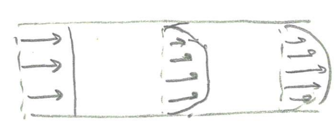
There are some cases called "nearly unidirectional flows" where we can make progress, e.g: nearly parallel plates

we don't have time to cover the "lubrication approximation" in §7.6 for these.

(2) Entrance / Edge Effects.

These effects occur for 2D or 3D cases, and involve flow in more than one direction,

e.g.



development of laminar flow in a tube.

↑ need v_z in order for v_r to change.

(3) Flow instabilities

Because our PDE is non-linear, there can be more than one solution. The solution from unidirectional flow may not be stable to a small disturbance. For example: laminar pipe flow at high Re becomes turbulent

* instabilities often, but don't always trigger turbulence.

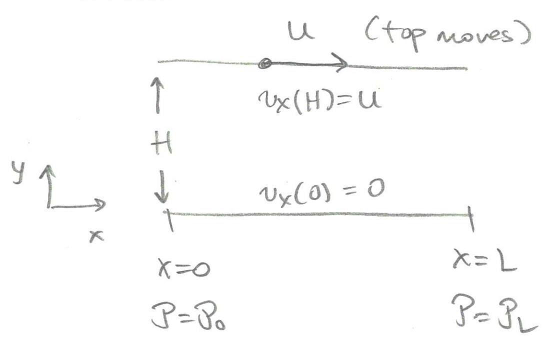
We could say much more about this subject, too, but we won't. Another study of a lifetime.

* what can we solve with unidirectional flow?

- pressure-driven flows (Poiseuille)
  - steady in a channel or tube
  - unsteady in a channel or tube
- boundary-driven flows (Couette)
  - steady in a channel or circle / annulus
  - unsteady near a plate or circle / annulus
- gravity-driven flow

II. Examples

A. Plane Couette & Poiseuille Flow (both together!)



- let $\mathcal{P}_0 > \mathcal{P}_L$, pressure-driven flow, too
- assume unidirectional, $v_x = v_x(y)$
- assume steady

Navier Stokes: $$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla\mathcal{P} + \mu\nabla^2\underline{v}$$

Simplifies to: (see table 6-7 for Cartesian)

(i) $$0 = -\frac{\partial \mathcal{P}}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2}$$ (x-momentum)

(ii) $$0 = -\frac{\partial \mathcal{P}}{\partial y}$$ (y-momentum)

(iii) $$0 = -\frac{\partial \mathcal{P}}{\partial z}$$ (z-momentum)

- (ii) & (iii) imply that $\mathcal{P} = \mathcal{P}(x)$ only
- using (i): $$\frac{\partial \mathcal{P}}{\partial x} = \mu\frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial P}{\partial x} = \text{constant} \rightarrow \text{why?}$$

$$= \frac{\mathcal{P}_L - \mathcal{P}_0}{L} = \frac{\Delta \mathcal{P}}{L}$$

only depends on x

$$\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{P}}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 v_x}{\partial y^2}\right) \neq 0 \quad \text{in general unless...}$$

Only way they are both equal then is if equal to a constant

- Now, solve for v_x

$$\mu \frac{d^2 v_x}{dy^2} = \frac{\Delta \mathcal{P}}{L} \xrightarrow[\text{2x}]{\text{integrate}} v_x = \frac{\Delta \mathcal{P}}{2\mu L} y^2 + c_1 y + c_2$$

total derivative b/c $v_x = v_x(y)$

- Apply BC's:

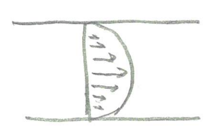
$$v_x(0) = c_2 = 0$$

$$v_x(H) = \frac{\Delta \mathcal{P}}{2\mu L} H^2 + c_1 H = U \Rightarrow c_1 = \frac{U}{H} - \frac{\Delta \mathcal{P} H}{2\mu L}$$

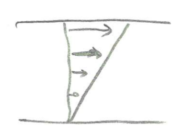
- Put all together:

$$v_x = \frac{\Delta \mathcal{P}}{2\mu L} y^2 + \frac{U}{H} y - \frac{\Delta \mathcal{P} H y}{2\mu L}$$

$$\boxed{v_x = -\frac{\Delta \mathcal{P}}{L}\frac{H^2}{2\mu}\left(\frac{y}{H} - \frac{y^2}{H^2}\right) + \frac{U y}{H}}$$

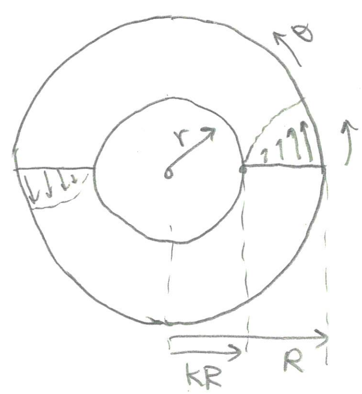


Poiseuille Contribution



Couette contribution

B. Couette Flow in an Annulus (Couette viscometer)



angular velocity, Ω_0

$v_\theta(R) = \Omega_0 R$

$v_\theta(KR) = 0$

no slip on inside & outside of annulus.

- assume unidirectional

$v_\theta = v_\theta(r)$ only

- assume steady

Navier-Stokes in cylindrical coords (table 6-8 on p. 240) (θ-momentum only):

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

- with $v_\theta = v_\theta(r)$ only and $v_r = v_z = 0$, this simplifies to:

$$0 = \underbrace{-\frac{1}{r}\frac{\partial P}{\partial \theta}} + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

This must be zero, because $P \neq P(\theta)$. It goes around in a circle. So no gradients possible in θ.

- Our PDE becomes the ODE:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r v_\theta)\right) = 0 \qquad v_\theta(R) = \Omega_0 R, \quad v_\theta(KR) = 0$$

- Now, integrate:

$$\frac{1}{r}\frac{d}{dr}(r v_\theta) = c_1 \Rightarrow \frac{d}{dr}(r v_\theta) = c_1 r$$

$$r v_\theta = \tfrac{1}{2}c_1 r^2 + c_2 \Rightarrow v_\theta = \tfrac{1}{2}c_1 r + c_2/r$$

- Apply BC's:

$$v_\theta(R) = \frac{c_1 R}{2} + \frac{c_2}{R} = \Omega_0 R \qquad (a)$$

$$v_\theta(KR) = \frac{c_1 K R}{2} + \frac{c_2}{KR} = 0 \qquad (b)$$

(A bit of algebra)

- multiply (b) by K & subtract (a) − (b):

$$\frac{c_1 R}{2} + \cancel{\frac{c_2}{R}} - \frac{c_1 K^2 R}{2} - \cancel{\frac{c_2}{R}} = \Omega_0 R$$

$$\frac{c_1 R}{2}(1-k^2) = \Omega_0 R \Rightarrow c_1 = \frac{2\Omega_0}{1-k^2}$$

– multiply (b) by $1/k$ and subtract (a)−(b):

$$\cancel{\frac{c_1 R}{2}} + \frac{c_2}{R} - \cancel{\frac{c_1 R}{2}} - \frac{c_2}{k^2 R} = \Omega_0 R$$

$$\frac{c_2}{R}\left(1-\frac{1}{k^2}\right) = \Omega_0 R \Rightarrow c_2 = \frac{\Omega_0 R^2}{1-1/k^2}$$

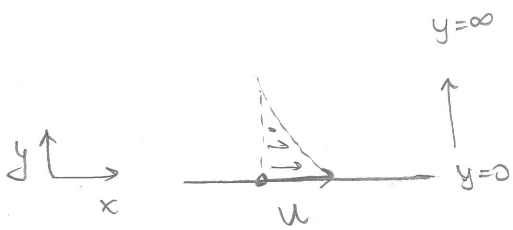
$$c_2 = \frac{\Omega_0 R^2 k^2}{k^2-1}$$

- Substitute into $v_\theta(r)$:

$$v_\theta = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\Omega_0}{1-k^2} r + \frac{\Omega_0 R^2}{k^2-1}\frac{1}{r}$$

$$\boxed{v_\theta = \frac{\Omega_0 k R}{1-k^2}\left(\frac{r}{kR} - \frac{kR}{r}\right)}$$

C. Startup of Couette Flow (If time)



- Fluid at rest before $t=0$
- Plate on bottom starts to move at u at $t=0$
- Assume unidirectional $v_x = v_x(y,t)$ only
- Unsteady

BC's $\begin{cases} v_x(0,t) = u \\ v_x(\infty,t) = 0 \end{cases}$

IC $\{ v_x(y,0) = 0$

(Table 6-7 x-momentum only)

* Navier Stokes in Cartesian Coords is:

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]$$

- $v_x = v_x(y,t)$, $v_y = v_z = 0$
- $\frac{\partial P}{\partial x} = 0$, no change along x-dir for P.

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2} \Rightarrow \frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \qquad \nu = \frac{\mu}{\rho}$$

* This is exactly the same as the transient diffusion problem we did with the similarity method!

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \qquad \begin{aligned} c(0,t) &= c_0 \\ c(\infty,t) &= 0 \\ c(x,0) &= 0 \end{aligned}$$

- In this case, we found that

$$\eta = \frac{x}{2\sqrt{Dt}} \quad \& \quad \frac{c}{c_0} = 1 - \mathrm{erf}(\eta)$$

* By analogy:

$$\boxed{\eta = \frac{y}{2\sqrt{\nu t}}} \quad \boxed{\frac{v_x}{u} = 1 - \mathrm{erf}(\eta)}$$

- ν, the kinematic viscosity is the momentum diffusivity.