# Lecture 17 - Creeping Flow

## I. Overview of creeping flow

* From a dimensional analysis of the Navier-Stokes equation, we found that at low Reynolds number, we get Stokes Equation:

$$0 = -\underline{\nabla} P + \mu \nabla^2 \underline{v}$$

* Sometimes we have additional body forces (as we will talk about shortly) and we can add those with an additional force term:

$$0 = -\underline{\nabla} P + \mu \nabla^2 \underline{v} + \underline{F}$$

* Sometimes you hear that turbulent flow is the most practical regime to study in fluid mechanics (often by those who study it, cough, Dr Lignell :) ). However, stokes flow or creeping flow is extremely useful & has been a focus of study in chemical engineering for the last 30 years.

* Creeping flow occurs at low Re, when lengths are small, viscosity is large, and flow is slow:

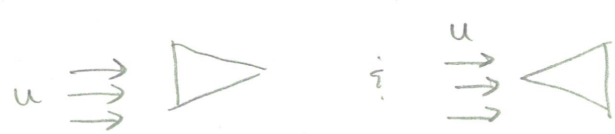
  - Oil recovery (viscous, small pores in rocks)
  - Biology (blood vessels, cells are small)

- colloids = paints, cosmetics, personal care products, consumer products
  (small nm to $\mu$m-size particles / suspensions)
- polymers = plastics, biopolymers (DNA, proteins)

Just to name a few cases of interest...

* The qualitative features of creeping flow are <u>not</u> <u>intuitive</u>:

- <u>No inertia</u>: (acceleration term is gone), aka "over-damped". When you stop the force, the fluid immediately stops.
  (our intuition is of fluids w/ inertia & turbulence).
  Think of swimming in Karo Syrup.
- <u>Pseudo-steady</u>: No time on RHS! As soon as apply a force, you get steady-state.
- <u>Reversible</u>: Stream lines are the same if the flow direction is reversed. This has odd consequences.
  - Fluid Mech video (stirring)
  - "stream lined" shapes don't matter



(like $u \leftrightarrow -u$)

Have the same drag force!

(Not same as  $u \neq -u$)

* Mathematically, the stokes Equation is linear.

  - The $\underline{v} \cdot \underline{\nabla} \underline{v}$ term gone!
  - This means solutions are unique! we can use superposition! The methods of ch. 5 apply!
  - Reversibility is a consequence of linearity

* we can re-write the Stokes equation in a way that is easier to solve. Doing so involves the stream function.

$$\underline{v} = \underline{\nabla} \psi \times \frac{\underline{e}_3}{h_3}$$

← unit vector in "3rd" dimension that you don't care about

← scale factor.

- example: cartesian

$$v_x = \frac{\partial \psi}{\partial y} \quad , \quad v_y = -\frac{\partial \psi}{\partial x}$$

See table 6-12 in Deen for more

- example: Cylindrical $(z, r)$, ie. $\theta$ symmetric

$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \quad , \quad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

- why this definition? Defined so that

$$\underline{\nabla} \cdot \underline{v} = \underline{\nabla} \cdot \left( \underline{\nabla} \psi \times \frac{\underline{e}_3}{h_3} \right) = 0$$

for any $\psi$. (Already accounts for continuity!)

- Especially useful for bidirectional flows (e.g. bidirectional or axisymmetric)

- The stream function is not the vorticity

$$\underline{\omega} = \underline{\nabla} \times \underline{v} \qquad \underline{\omega}: \text{vector}$$

$$\underline{v} = \nabla\psi \times \left(\frac{\underline{e}_3}{h_3}\right) \qquad \psi: \text{scalar}$$

But, you can re-write Navier stokes in terms of $\psi$ & $\underline{\omega}$. why do this? Because we don't have an explicit expression for P. So, we change variables.

Example: Cartesian, $\underline{e}_3 = \underline{e}_z$

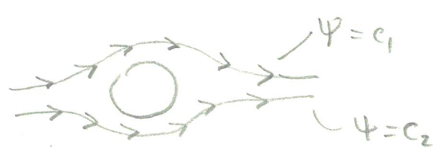
$$\nabla^2\psi = -\omega_z$$

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\nabla}\,\underline{v} + \nu \nabla^2 \underline{\omega}$$

vorticity equation (take curl of N-S)

- what does the stream function mean?

Constant values of $\psi$ are "streamlines". They are parallel /tangent to velocity vectors




* For stokes Equation, the vorticity equation & stream function become more simple:

How get these?
(1) $\nabla \times$ (N-S) $\rightarrow$ vorticity Eq.
(2) plug into $\nabla^2\psi = \omega_z$.

$$\left.\begin{aligned} &\nabla^2 \underline{\omega} = 0 \\ &\boxed{\nabla^4 \psi = 0} \end{aligned}\right\} \text{Cartesian}$$

$$\left.\begin{aligned} &\nabla^2 \underline{\omega} = 0 \\ &\boxed{E^4 \psi = 0} \end{aligned}\right\} \text{axisymmetric}$$

"biharmonic Eq."
"stream function form" of stokes Eq.

$$\nabla^4 = \nabla^2(\nabla^2) = \nabla\cdot\nabla(\nabla\cdot\nabla)$$ → biharmoic operator / biLaplacian operator

$$E^4 = E^2(E^2)$$

recall: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (cartesian, 2D)

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad \text{(cylindrical, } rz\text{)}$$

↑ This is a new operator. It is "made-up" here for this, so we don't have to write out more stuff!

See Table 6-12 in §6.8.

* we "only" need to solve

$$\nabla^4\psi = 0 \quad \text{or} \quad E^4\psi = 0$$

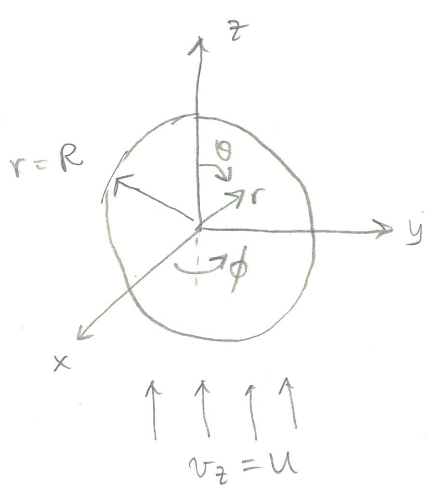
in order to find $\underline{v}$! we got rid of continuity!

This is linear! (4th order PDE)

II. Example: Flow around a sphere. (Example 8.4-2)

* this is an important example. Colloids are often approximated as spheres. This solution forms the basis for colloidal hydrodynamics!

$\boxed{\text{Start problem. Finish for HW}}$




Stokes Eq: $E^4\psi = 0$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

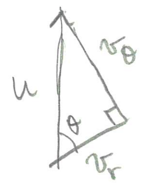
- spherical coordinates
- $\phi$-symmetric

$\Rightarrow \psi(r,\theta)$

* what are the boundary conditions?

$$v_r(R,\theta) = 0 \quad , \quad v_\theta(R,\theta) = 0$$

no penetration/ no slip on surface




$$v_r(\infty,\theta) \to u\cos\theta$$

$$v_\theta(\infty,\theta) \to -u\sin\theta$$

- Re-write using definition of stream function

$$v_r(R,\theta) = \boxed{0 = \frac{\partial\psi}{\partial\theta}(R,\theta)}$$

$$v_\theta(R,\theta) = \boxed{0 = \frac{\partial\psi}{\partial r}(R,\theta)}$$

$$v_r(\infty,\theta) = u\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}(\infty,\theta)$$

Show this

$$\frac{\partial\psi}{\partial\theta}(\infty,\theta) = r^2 u\sin\theta\cos\theta$$

$$v_\theta(\infty,\theta) = -u\sin\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}(\infty,\theta)$$

$$\frac{\partial\psi}{\partial r}(\infty,\theta) = -ru\sin^2\theta$$

Integrate : $\boxed{\psi(\infty,\theta) = \dfrac{r^2 u\sin^2\theta}{2}} + C$

$\frac{\partial}{\partial\theta}\sin^2\theta = 2\sin\theta\cos\theta$

$\frac{\partial}{\partial r} r^2 = 2r$

set to 0, b/c $\psi$ is defined to w/in arbitrary constant.

* Now, we have BC's and we can proceed.

~~• Based on the BC at infinity and our intuition from PDEs we guess a product solution:~~

* Now, we proceed to solve via an FFT-like method.

$$C_n(r) = \int_a^b \psi(r,\theta)\, D_n(\theta)\, w(\theta)\, d\theta$$

eigen functions (use $D_n$ instead of $\psi_n$ to avoid confusion)

$$\psi(r,\theta) = \sum_{n=1}^{\infty} C_n(r)\, D_n(\theta)$$

- what are the eigenfunctions of $E^4$?

  Based on §5.8, we expect them to be in the $\theta$ direction ($r$ is not homogeneous). However, we don't know what they are!

- Our boundary condition at $r \to \infty$ is of the form $f(r) \sin^2\theta$. Lets guess that $D_n(\theta) = \sin^2\theta$.

  - This is only one eigenfunction. However, if our solution is "along" an eigenfunction, we only need one.

  - vector-function analogy, if our answer is along $e_x$, then we only need one component, even in $\mathbb{R}^3$.

  - This is different than ch. 5 because we have a special B.C. It is not constant and it is an eigenfunction.

* Substitute $\psi(r,\theta) = f(r) \sin^2\theta$ into $E^4\psi = 0$.

  This is a proof that $\sin^2\theta$ is an eigen function:

Don't do in class.
Too long

$$E^4\psi = 0 = E^2(E^2\psi) = 0$$

$$= E^2\left[\frac{\partial^2}{\partial r^2}(f\sin^2\theta) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(f\sin^2\theta)\right)\right]$$

$$= E^2\left[\sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{f\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{2\sin\theta\cos\theta}{\sin\theta}\right)\right]$$

$$= E^2\left[\sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2f\sin\theta}{r^2}\sin\theta\right]$$

$$= E^2\left[\sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\sin^2\theta\right]$$

$$= \frac{\partial^2}{\partial r^2}\left[\sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\sin^2\theta\right]$$

$$+ \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\sin^2\theta\right)\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right)$$

$$+ \frac{\partial^2 f}{\partial r^2}\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right)$$

$$- \frac{2f\sin\theta}{r^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right)$$

$$+ \left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right)\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right)}_{\underbrace{\frac{\partial}{\partial\theta}\left(\frac{2\sin\theta\cos\theta}{\sin\theta}\right)}_{-2\sin\theta}}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right)$$

$$-2\frac{\sin^2\theta}{r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right)$$

$$= \sin^2\theta \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right) f = 0$$

show this

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)^2 f = 0}$$

* The above got rid of $\theta$-dependence, proving $\sin^2\theta$ is an eigenfunction.

* Now we need to solve the ODE for $f$. (The "$c_n(r)$" in the FFT.)

• Expand the differential equation:

skip this

$$\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right) - \frac{2}{r^2}\left(\frac{\partial^2 f}{\partial r^2} - \frac{2f}{r^2}\right) = 0$$

$$\Rightarrow \frac{\partial^4 f}{\partial r^4} - \underbrace{\frac{\partial^2}{\partial r^2}\left(\frac{2f}{r^2}\right)}_{} - \frac{2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{4f}{r^4} = 0$$

$$\underbrace{2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial f}{\partial r} + \frac{-2}{r^3} f\right)}_{}$$

$$2\left[\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{r^3}\frac{\partial f}{\partial r} - \frac{2}{r^3}\frac{\partial f}{\partial r} + \frac{6}{r^4} f\right]$$

$$\Rightarrow \frac{\partial^4 f}{\partial r^4} - \frac{2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{4}{r^3}\frac{\partial f}{\partial r} + \frac{4}{r^3}\frac{\partial f}{\partial r} - \frac{12}{r^4} f$$

$$-\frac{2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{4f}{r^4} = 0$$

$$\Rightarrow \frac{\partial^4 f}{\partial r^4} - \frac{4}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{8}{r^3}\frac{\partial f}{\partial r} - \frac{8}{r^4} f = 0$$

$$\Rightarrow \boxed{r^4 \frac{\partial^4 f}{\partial r^4} - 4r^2\frac{\partial^2 f}{\partial r^2} + 8r\frac{\partial f}{\partial r} - 8f = 0}$$

say this

* This is an equidimensional equation. See § B.5. Homogeneous solutions are of the form $Cr^m$.

* Substitute $Cr^m$ into above equation will give a characteristic equation, roots give the solution.

Stop here

Do rest for HW

$$r^4 \cdot \frac{\partial^4}{\partial r^4}(r^m) - 4r^2 \frac{\partial^2}{\partial r^2}(r^m) + 8r \frac{\partial}{\partial r}(r^m) - 8r^m = 0$$

$$r^4 \cdot m \cdot (m-1) \cdot (m-2)(m-3) r^{m-4} - 4r^2 m(m-1) r^{m-2}$$
$$+ 8 r m r^{m-1} - 8r^m = 0$$

$$m \cdot (m-1)(m-2)(m-3)\cancel{r^m} - 4m(m-1)\cancel{r^m} + 8m\cancel{r^m} - 8\cancel{r^m} = 0$$

$$\Rightarrow m(m-1)(m-2)(m-3) - 4m(m-1) + 8(m-1) = 0$$

roots? $(m-1)$ is a factor $\Rightarrow$ $\boxed{m=1}$

$$\Rightarrow m(m-2)(m-3) \underbrace{-4m+8}_{-4(m-2)} = 0$$

$$\Rightarrow m(m-2)(m-3) - 4(m-2) = 0$$

$m-2$ is a factor $\Rightarrow$ $\boxed{m=2}$

$$\Rightarrow m(m-3) - 4 = 0$$

$$\Rightarrow m^2 - 3m - 4 = 0$$

$$\Rightarrow (m+1)(m-4) = 0 \qquad \boxed{\begin{matrix} m=4 \\ m=-1 \end{matrix}}$$

* Solutions are of the form:

$$\boxed{f(r) = Ar^4 + Br^2 + Cr + Dr^{-1}}$$

* Now, apply the boundary conditions to solve for A, B, C, & D.

$$\frac{\partial \psi}{\partial \theta}(R,\theta) = 0 = \frac{\partial}{\partial \theta}\left(f(R)\sin^2\theta\right) = f(R)\frac{\partial}{\partial \theta}\left(\sin^2\theta\right) = 0$$

$$\boxed{f(R) = 0}$$

$$\frac{\partial \psi}{\partial r}(R,\theta) = 0 = \frac{\partial}{\partial r}\left(f(r)\sin^2\theta\right)\Big|_{r=R} = \sin^2\theta\,\frac{\partial f}{\partial r}(R) = 0$$

$$\boxed{\frac{\partial f}{\partial r}(R) = 0}$$

$$\lim_{r\to\infty}\psi(r,\theta) = \frac{r^2 U\sin^2\theta}{2} = \lim_{r\to\infty} f(r)\sin^2\theta$$

$$\boxed{\lim_{r\to\infty} f(r,\theta) = Ur^2/2}$$

* Apply BC's to find A, B, C, D:

$$\lim_{r\to\infty} f(r) = Ar^4 = \frac{Ur^2}{2} \Rightarrow \boxed{A = 0}$$ or else $r^4$ would dominate.

$$= Br^2 = \frac{Ur^2}{2} \Rightarrow \boxed{B = U/2}$$

$$\begin{cases} f(R) = AR^4 + BR^2 + CR + D/R = 0 \\ \dfrac{df}{dr}(R) = 4AR^3 + 2BR + C - D/R^2 = 0 \end{cases}$$

$$\to f(R) = \frac{u}{2}R^2 + CR + \frac{D}{R} = 0 \quad (1)$$

$$\frac{df}{dr}(R) = uR + C - D/R^2 = 0 \quad (2) \quad = uR^2 + CR - D/R$$

mult. (2) by R and subtract/add from (1):

$$-\frac{u}{2}R^2 + \frac{2D}{R} = 0 \qquad D = \frac{u}{4}uR^3$$

$$\frac{3}{2}uR^2 + 2CR = 0 \qquad C = -\frac{3}{4}uR$$

$$f(r) = \frac{u}{2}r^2 - \frac{3}{4}uRr + \frac{u}{4}uR^3/r$$

$$= uR^2\left[\frac{1}{2}\frac{r^2}{R^2} - \frac{3}{4}\frac{r}{R} + \frac{1}{4}\frac{R}{r}\right]$$

$$\boxed{\psi(r,\theta) = uR^2\sin^2\theta\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{3}{4}\left(\frac{r}{R}\right) + \frac{1}{4}\left(R/r\right)\right]}$$

* Now, get velocity components:

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{2\cancel{\sin\theta}\cos\theta \cdot uR^2}{r^2\cancel{\sin\theta}}\left[\frac{1}{2}\frac{r^2}{R^2} - \frac{3}{4}\frac{r}{R} + \frac{1}{4}\frac{R}{r}\right]$$

$$\boxed{v_r = u\cos\theta\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\frac{R^3}{r^3}\right]}$$

$$v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \frac{-uR^2\sin^2\theta}{r\sin\theta}\left[\frac{r}{R^2} - \frac{3}{4}\frac{1}{R} - \frac{1}{4}\frac{R}{r^2}\right]$$

$$\boxed{v_\theta = -U \sin\theta \left[ 1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\frac{R^3}{r^3} \right]}$$

* $p(r,\theta)$ can be obtained from integrating Stokes Equation

$$\boxed{p(r,\theta) = -\frac{3}{2}\frac{\mu U}{R}\left(\frac{R}{r}\right)^2 \cos\theta}$$

Drag is famous!

$$\boxed{F_D = 6\pi\mu U R}$$

## III. Example: The stokeslet.

* we desire the solution to :

$$0 = -\nabla p + \mu\nabla^2 \underline{v} + \underline{F}$$

where $\underline{F} = \delta(\underline{r})\, \underline{f}$ ← constant in space.

* One solution? velocity disturbance :

$$\underline{v} - \underline{v}_\infty = \begin{cases} v_r = U\cos\theta \left[ -3/2\, R/r + 1/2\, R^3/r^3 \right] \\ v_\theta = -U\sin\theta \left[ -3/4\, R/r - 1/4\, R^3/r^3 \right] \end{cases}$$

- from drag, let $F_z = -F_D$ : force from sphere on fluid

$$R = \frac{-F_z}{6\pi\mu U}$$

- Consider $R \to 0$ (or far away from $R$), we look like a point force.

$$R^3 << R \to \text{neglect } R^3$$

$$v_r = U\cos\theta \left( +3/2 \cdot \frac{+F_z}{6\pi\mu U r} \right) = \frac{F_z}{4\pi\mu r}\cos\theta$$

$$v_\theta = -U\sin\theta \left( +\frac{3}{4} \cdot \frac{+F_z}{6\pi\mu U r} \right) = \frac{-F_z}{8\pi\mu r}\sin\theta$$

$$\boxed{\underline{v}(r,\theta) = \frac{F_z}{8\pi\mu r}\left(2\cos\theta\, \underline{e}_r - \sin\theta\, \underline{e}_\theta\right)}$$

* we can solve this more generally* with a Fourier Transform (not finite, regular one).

* for all directions

let $\hat{f}(\underline{k}) = \mathcal{F}[f(\underline{r})] = \int_{-\infty}^{\infty} f(\underline{r})\, e^{-i\underline{k}\cdot\underline{r}}\, d\underline{r}$ F.T.

$f(\underline{r}) = \mathcal{F}^{-1}[\hat{f}(\underline{k})] = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} f(\underline{k})\, e^{+i\underline{k}\cdot\underline{r}}\, d\underline{r}$ I.F.T.

- Note: $\mathcal{F}[\nabla^2 f] = -k^2\hat{f}$

  $\mathcal{F}[\nabla f] = i\underline{k}\hat{f}$

  imaginary number : $i = \sqrt{-1}$

  $\mathcal{F}[\delta(\underline{r})] = 1$

* Now, solve with Fourier Transform

- Take F.T. of Stokes

$$\mathcal{F}[0] = \mathcal{F}[-\nabla p + \mu\nabla^2\underline{v} + \delta\underline{f}]$$

$$0 = -i\underline{k}\hat{p} - \mu k^2\hat{\underline{v}} + \underline{f}$$

- Take divergence of Stokes.

$$\underline{\nabla}\cdot 0 = \underline{\nabla}\cdot\left[-\underline{\nabla}p + \mu\nabla^2\underline{v} + \delta\underline{f}\right]$$

$$= -\nabla^2 p + \mu\nabla^2(\cancel{\underline{\nabla}\cdot\underline{v}}) + \underline{\nabla}\cdot\delta\underline{f}$$

0

continuity

$$\nabla^2 p = \underline{\nabla}\cdot(\delta\underline{f})$$

- Take F.T. of pressure-Poisson Eq:

$$\mathcal{F}[\nabla^2 p] = \mathcal{F}[\underline{\nabla} \cdot (\delta \underline{f})]$$

$$-k^2 \hat{p} = i\underline{k} \cdot \underline{f}$$

$$\boxed{\hat{p} = \frac{-i\underline{k} \cdot \underline{f}}{k^2}}$$

- Plug into F.T. of Stokes.

$$0 = -i\underline{k}\frac{(-i\underline{k} \cdot \underline{f})}{k^2} - \mu k^2 \hat{\underline{v}} + \underline{f}$$

$$\mu k^2 \hat{\underline{v}} = -\frac{\underline{k}\underline{k} \cdot \underline{f}}{k^2} + \underline{f}$$

$$\boxed{\hat{\underline{v}} = \frac{1}{\mu k^2}\left(-\frac{\underline{k}\underline{k}}{k^2} + \underline{\underline{\delta}}\right) \cdot \underline{f}}$$

3 terms: $\mathcal{F}^{-1}\left[\frac{i\underline{k}}{k^2}\right]$, $\mathcal{F}^{-1}\left[\frac{1}{k^2}\right]$, $\mathcal{F}^{-1}\left[\frac{\underline{k}\underline{k}}{k^4}\right]$

- Take the inverse F.T. (this is laborious)

See notes by Lisicki

$$\boxed{p(\underline{r}) = \frac{\underline{r} \cdot \underline{f}}{4\pi r^3}}$$

$\underline{r} = [x, y, z]^T$ in Cartesian

$r = |\underline{r}|$

$$\boxed{\underline{v}(\underline{r}) = \frac{1}{8\pi\mu r}\left(\frac{\underline{r}\underline{r}}{r^2} + \underline{\underline{\delta}}\right) \cdot \underline{f}}$$

- Substitute $\underline{f} = [0, 0, f_z]$ and spherical coordinates for $\underline{r}$.

(compare to earlier)

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$$\underline{\underline{\delta}} \cdot \underline{f} = \underline{f} = f_z \underline{e}_z$$

$$\underline{r} \cdot \underline{f} = r\cos\theta \, f_z$$

$$\underline{r}\,\underline{r}\cdot\underline{f} = \underline{r}(\underline{r}\cdot\underline{f}) = \underline{r}\,(r\cos\theta\, f_z) = r^2\cos\theta\, f_z\, \underline{e}_r$$

$$\to \underline{r} = r\underline{e}_r$$

So,
$$\boxed{p(\underline{r}) = \frac{f_z\cos\theta}{4\pi r^2}}$$

$$\underline{v}(\underline{r}) = \frac{1}{8\pi\mu r}\left(\frac{\cancel{r^2}\cos\theta\, f_z}{\cancel{r^2}}\underline{e}_r + f_z\,\underline{e}_z\right)$$

$$\underline{e}_z = \cos\theta\,\underline{e}_r - \sin\theta\,\underline{e}_\theta$$

$$\underline{v}(r) = \frac{1}{8\pi\mu r}\left(\cos\theta\, f_z\,\underline{e}_r + f_z\cos\theta\,\underline{e}_r - f_z\sin\theta\,\underline{e}_\theta\right)$$

$$\boxed{\underline{v}(r) = \frac{f_z}{8\pi\mu r}\left(2\cos\theta\,\underline{e}_r - \sin\theta\,\underline{e}_\theta\right)}$$

Same as on page 10!

* Comments:

- The solution with a delta function is a special solution called a "Green's function." we can add up many of them to get a more complex solution. Superposition! (fundamental solution)
- The stokeslet is the "far-field" solution for a sphere. Accurate far away from $r = R$.
- we can write $\underline{v} = \underline{\underline{\Lambda}}\cdot\underline{f}$

$$\underline{\underline{\Lambda}} = \frac{1}{8\pi\mu r}\left[\frac{\underline{r}\,\underline{r}}{r^2} + \underline{\underline{\delta}}\right]$$ is called the Oseen Tensor.