Lecture 19 - Boundary Layer Theory

I. D'Alembert's Paradox

* Last time we saw that there was a problem with our solution for potential flow over a cylinder: it doesn't satisfy no-slip!
  - No drag force! No friction drag & no pressure drag.
  - What about turbulence? No vorticity ever? Even at super high Re?

* We are missing something important. Let's go back to our dimensional analysis of Navier-Stokes. We said at high Re:

$$\frac{D\tilde{\underline{v}}}{Dt} = -\tilde{\nabla} p + \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{v}} \qquad , Re \gg 1$$

  - Think about this mathematically. This is a 2^{nd} order PDE in space. What happens when $Re \to \infty$? We get a 1^{st} order PDE (in space).

    2^{nd} order DE $\rightarrow$ 2 BCs
    1^{st} " " $\rightarrow$ 1 BC.

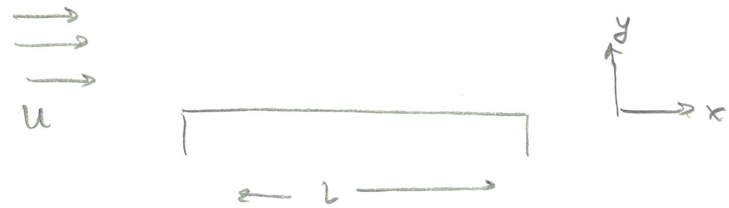
  - We did something weird.
  - Mathematically, we did what is called a "singular perturbation" (back in §4.4)
  - Physically, we neglected the fact that very close to the cylinder surface, δ.

$$Re = \frac{\rho U \delta}{R} \Rightarrow Re \text{ is not big if } \delta \text{ is small!}$$

- we don't have time to go over singular perturbation problems :(. Maybe next year.

II. Boundary Layer Equations

* let's look at an "easier" problem than the cylinder. what is the flow profile above a flat plate.



can check $\nabla^2 \phi = 0$ ✓

- The potential flow solution is trivial.

$$\boxed{v_x = u}$$ ← also doesn't satisfy no slip.

* let's do a more careful dimensional analysis close to the flat plate.

- 2D, cartesian coordinates, steady, $Re_L = \frac{\rho u L}{\mu} \gg 1$

(continuity) $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

(x-momentum) $$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

(y-momentum) $$\rho \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

* Now, let's pick scales for dimensional analysis

$$\tilde{x} = x/L \qquad \tilde{v}_x = v_x/u \qquad \tilde{P} = P/\rho u^2$$

$$\tilde{y} = y/\delta \qquad \tilde{v}_y = v_y/V$$

new scale for y
don't assume anything

new scale for y
velocity

- continuity:

$$\frac{u}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

$$\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{LV}{u\delta}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$$

- The π-theorem still only gives us one D.O.F., Re_L. So, this group must be 1.

$$\frac{LV}{u\delta} = 1 \Rightarrow \boxed{\frac{u}{L} = \frac{V}{\delta}}$$

- Continuity says the velocity gradients have to be on the same order:

if $\frac{\Delta v_x}{\Delta x} \approx \frac{u}{L}$ then $\frac{\Delta v_y}{\Delta y} \approx \frac{V}{\delta}$

↳ mass can't disappear, so if v_x slows down, must have v_y

- x-momentum:

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\Rightarrow \rho\left(\frac{U^2}{L}\tilde{v}_x\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{VU}{\delta}\tilde{v}_y\frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right) =$$

$$-\frac{\rho U^2}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu\left(\frac{U}{L^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{U}{\delta^2}\frac{\partial^2 v_x}{\partial \tilde{y}^2}\right)$$

- Note: $\frac{VU}{\delta} = \frac{U}{\delta}\frac{U\delta}{L} = \frac{U^2}{L}$

$$\Rightarrow \frac{\rho U^2}{L}\left(\tilde{v}_x\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right) = -\frac{\rho U^2}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\mu U}{L^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}$$

$$+ \frac{\mu U}{\delta^2}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

$$\Rightarrow \tilde{v}_x\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_x}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \underbrace{\frac{\mu}{\rho U L}}_{Re_L^{-1}}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \underbrace{\frac{\mu L}{\rho U \delta^2}}_{?}\frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

- Up until now, everything is the same as when we non-dimensionalized N-S before except the last term on the RHS.
- we have 2 choices: (a) let $\delta = L$. Same as before. (b) let $(?) = 1$. Then derivative doesn't go away! No longer singular. Can satisfy no-slip!

why not let it be something else? No other D.O.F. Can't have another dimensionless group.

$$\frac{\mu L}{\rho U \delta^2} = 1 \Rightarrow \frac{\mu}{\rho U L}\cdot\frac{L^2}{\delta^2} = 1$$

$$\Rightarrow \frac{1}{Re_L}\frac{L^2}{\delta^2} = 1$$

$$\boxed{\frac{\delta}{L} \sim Re_L^{-1/2}}$$

or

$$\boxed{\delta \sim \left(\frac{\mu L}{\rho U}\right)^{1/2} \sim \left(\frac{\nu L}{U}\right)^{1/2}}$$

* Comments:

- δ is a new length scale. It is a boundary layer height.

- δ is the length scale where viscous stress still matters. $\frac{\partial^2 v_x}{\partial y^2}$ gets so big as we get close to the plate that we can't ignore it.

- There are two "regions" for this flow.
  - Outer region (far from plate):
    potential flow / inviscid flow is good.
    $v_x = U$ is solution
  - Inner region (close to plate):
    need above equations.

- when $Re_L \gg 1$, $\delta/L \ll 1$. The boundary layer is small.

- we "re-scaled" to this inner region.

- In the limit that $Re_L \to \infty$, we get:

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = -\frac{\partial P}{\partial \tilde{x}} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}$$

OR

$$\boxed{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2}}$$

"Boundary Layer Equations"

- what about v_y?
  - y-momentum gives: $\frac{\partial p}{\partial y} = 0$ (sort of useless)
  - continuity gives: $v_y = -\int \frac{\partial v_x}{\partial x} dy$ (use this)

- This equation is actually more useful than you might think. Curved surfaces look flat when you get really close. One can show (see Gary Leal, Advanced Transport Phenomena, §10F) that curved geometries give the same boundary layer equation because $\delta/R \ll 1$.

III. Solving Boundary Layer Problems

* How do we solve for a flow at high Re and account for the boundary layer?

(1) Solve outer problem using potential flow. Use B.C. far away to find $\underline{v}$.

e.g. $\nabla^2 \phi = 0$, $\underline{v} = \underline{\nabla}\phi$, $\underline{v}(r \to \infty) = \underline{v}_\infty$

(2) Use Euler's equation to find pressure gradient

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla} p \Rightarrow \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$$

(3) Solve Boundary layer equations for inner region velocities.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0, \quad v_y = -\int \frac{\partial v_x}{\partial x} dy, \quad \underline{v}\big|_{surface} = \underline{0}$$

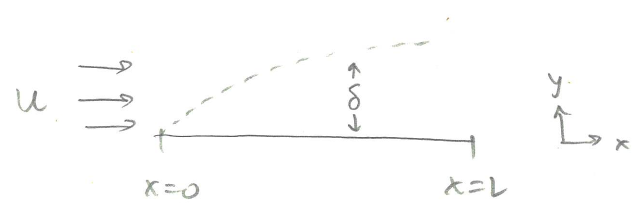
(4) Use no-slip (inner) BC and a matching condition to find final velocities.

$$\underline{v}\,\big|_{surface} = 0 \qquad \text{(no slip)}$$

$$\underline{v}(r \to \infty)\big|_{inner} = \underline{v}(r \to 0)\big|_{outer}$$

Combine (& subtract overlap if necessary) to find $\underline{v}$!

* Example: Flat Plate (Blasius Solution)



(1) Outer solution

$$\boxed{v_x = U, \quad v_y = 0} \qquad \text{(easy)}$$

(2) Find ∇P

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{\partial P}{\partial x} \qquad \text{(Euler's equation)}$$

$$v_x = U = \text{const} \Rightarrow \boxed{\frac{\partial P}{\partial x} = 0}$$

(3) Solve BL equations

$\frac{\partial P}{\partial x}$ is gone

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

- Non-linear PDE.
- Blasius tried a similarity transform

$$\eta = y/\delta(x) \qquad \delta(x) = \left(\frac{2\nu x}{U}\right)^{1/2}$$

- This is a good idea, the problem is semi-infinite.
- This looks similar to our semi-infinite diffusion problem.
- where did we get $\delta(x)$? From scaling arguments, $L \to x$, added a 2 for ease.
- Define the transform:

$$\boxed{\eta = y/\delta} \qquad \theta = v_x/u. \quad (*)$$

(*) Blasius didn't like θ. He wanted to use a stream function instead:

$$v_x = -\frac{\partial \psi}{\partial y} \qquad u\theta = -\frac{1}{\delta}\frac{\partial \psi}{\partial \eta}$$

$$f = \frac{-\psi}{u\delta} \Rightarrow \text{dimensionless stream function}$$

$$\boxed{\frac{v_x}{u} = \frac{\partial f}{\partial \eta}}$$

- Now, substitute into BL equations. This is a lot of algebra, See appendix.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2}$$

$$\left\{ \begin{array}{l} v_x = uf', \quad \dfrac{\partial v_x}{\partial x} = -\dfrac{\mu}{\rho}\dfrac{\eta}{\delta^2} f'', \quad v_y = \dfrac{\mu}{\rho\delta}\left[\eta f' - f\right] \\[2ex] \dfrac{\partial v_x}{\partial y} = \dfrac{u}{\delta} f'', \quad \dfrac{\partial^2 v_x}{\partial y^2} = \dfrac{u}{\delta^2} f''' \end{array} \right.$$

where $f' = \dfrac{df}{d\eta}$, $f'' = \dfrac{d^2 f}{d\eta^2}$, etc.

- Substituting gives:

$$Uf'\left(-\frac{\mu}{\rho}\frac{\eta}{\delta^2}f''\right) + \frac{\mu}{\rho\delta}\left[\eta f' - f\right]\frac{U}{\delta}f'' = \frac{\mu}{\rho}\frac{U}{\delta^2}f'''$$

$$-\frac{\mu U}{\rho}\cancel{\frac{\eta}{\delta^2}f'f''} + \frac{\mu U}{\rho}\cancel{\frac{\eta}{\delta^2}f'f''} - \frac{\mu U}{\rho}\frac{1}{\delta^2}ff'' = \frac{\mu U}{\rho}\frac{f'''}{\delta^2}$$

$$-\cancel{\frac{\mu U}{\rho}}\cancel{\frac{1}{\delta^2}}ff'' = \cancel{\frac{\mu U}{\rho}}\cancel{\frac{1}{\delta^2}}f'''$$

$$\boxed{f''' + ff'' = 0}$$ "Blasius Equation"

- Also need BC's:

(#4) →

$v_x(0) = 0$ (no slip) → $\boxed{f'(0) = 0}$

$v_y(0) = 0$ (no penetration) → $\boxed{f(0) = 0}$

$$\eta \underbrace{\cancel{f'}}_{0} - f = 0$$

$v_x(\infty) = U$ (match to outer soln) → $\boxed{f'(\infty) = 1}$

- well-posed, nonlinear ODE. No known analytical solution. Blasius tabulated a numerical solution. So can we!

  (see python code)

* Comments

- we solved for flat plate velocity profile.
- 4 steps can be used for other geometries. These change the $\partial P/\partial x$ term.
- These are laminar boundary layers. Turbulence is even more complex.