* Substituting these in:

$$U f' \cdot \left(-\eta \nu \frac{f''}{\delta^2}\right) + \frac{\nu}{\delta}(\eta f' - f)\left(\frac{U f''}{\delta}\right) = \nu \frac{U}{\delta^2} f'''$$

$$\cancel{-\frac{\eta \nu U}{\delta^2} f' f''} + \cancel{\frac{\nu U \eta}{\delta^2} f' f''} - \frac{\nu U}{\delta^2} f f'' = \frac{\nu U}{\delta^2} f'''$$

$$-\frac{\nu U}{\delta^2} f f'' = \frac{\nu U}{\delta^2} f'''$$

Supplemental notes for

Boundary Layers

* Now we have a well-posed, non-linear ODE for $f(\eta)$. How do we solve it?

<u>Numerically!</u> Blasius also tabulated his solution

2. Numerical Solution of Blasius Equation

* How solve:

$$\left.\begin{aligned} &f''' + f f'' = 0 \\ &f(0) = 0 \\ &f'(0) = 0 \\ &f'(\infty) = 1 \end{aligned}\right\} \quad \text{let} \quad \begin{aligned} z_0 &= f \\ z_1 &= f' \\ z_2 &= f'' \end{aligned}$$

* This turns our equations into:

$$\frac{dz_0}{d\eta} = z_1 \quad , \quad z_0(0)=0 \qquad \text{(Eq. for } f\text{)}$$

$$\frac{dz_1}{d\eta} = z_2 \quad , \quad z_1(0)=0 \qquad \text{(Eq. for } f'\text{)}$$

$$\frac{dz_2}{d\eta} = -z_0 z_2 \quad , \quad z_2(0)=c \qquad \text{(Eq. for } f''\text{)}$$

This was not our BC. It was $z_1(\infty)=1$. Solve ODEs using a guess for c. Then iterate & solve until:

$$z_1(\infty)=1$$

is satisfied by changing c.

* See python code & plot.

* For plotting, it is more convenient to use: scale for outer solution.

$$\tilde{v}_x = \frac{v_x}{u} \quad , \quad \tilde{v}_y = \frac{v_y}{u} \quad , \quad \tilde{x} = \frac{x}{L} \quad , \quad \tilde{y} = y/L \quad , \quad \tilde{\delta} = \delta/L$$

$$\rightarrow \eta = y/\delta = \tilde{y}/\tilde{\delta} = \tilde{\eta}$$

$$Re_L = \frac{Lu}{\nu}$$

- Relating these to f, f', f'':

$$v_x = uf' \Rightarrow \boxed{\tilde{v}_x = f'}$$

$$v_y = \frac{\nu}{\delta}[\eta f' - f] \Rightarrow \tilde{v}_y = \frac{\nu}{\delta u}[\eta f' - f]$$

$$= \frac{\nu}{\tilde{\delta} L u}[\tilde{\eta} f' - f]$$

$$\boxed{\tilde{v}_y = \frac{1}{\tilde{\delta} Re_L}[\tilde{\eta} f' - f]}$$

$$\delta = \sqrt{\frac{2\nu x}{u}} \Rightarrow \tilde{\delta} = \sqrt{\frac{2\nu x}{L^2 u}} = \sqrt{\frac{2\nu \tilde{x}}{Lu}} \quad , \quad \boxed{\tilde{\delta} = \left(\frac{2\tilde{x}}{Re_L}\right)^{1/2}}$$

Supplemental Notes

* Aside: y-momentum for BL equations

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

steady

$$\rho\left(\frac{UV}{L}\tilde{v}_x\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \frac{V^2}{\delta}\tilde{v}_y\frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right) = -\frac{\rho U^2}{\delta}\frac{\partial \tilde{P}}{\partial \tilde{y}} + \frac{\mu V}{L^2}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{\mu V}{\delta^2}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}$$

$$\frac{\rho U V}{L} = \frac{\rho U^2 \delta}{L^2} \qquad \rho\frac{V^2}{\delta} = \rho\frac{U^2\delta^2}{\delta L^2} = \frac{\rho U^2 \delta}{L^2}$$

$$\frac{\mu V}{\delta^2} = \frac{\mu U \delta}{\delta^2 L} = \frac{\mu U}{\delta L} \qquad \frac{\mu V}{L^2} = \frac{\mu U \delta}{L^3}$$

$$\tilde{v}_x\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = -\frac{L^2}{\delta\rho U^2}\cdot\frac{\rho U^2}{\delta}\frac{\partial \tilde{P}}{\partial \tilde{y}} + \frac{\cancel{L^2}}{\cancel{\delta}\rho U^2}\frac{\mu U \cancel{\delta}}{\cancel{L^3}\,L}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2}$$

$$+ \frac{L^2}{\delta\rho U^2}\frac{\mu U}{\delta L}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}$$

$$\tilde{v}_x\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = -\underbrace{\frac{L^2}{\delta^2}}_{\frac{L^2}{\delta^2} = Re_L}\frac{\partial \tilde{P}}{\partial \tilde{y}} + \underbrace{\frac{\mu}{\rho U L}}_{Re_L^{-1}}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \underbrace{\underbrace{\frac{\mu}{\rho U L}}_{Re_L^{-1}}\underbrace{\frac{L^2}{\delta^2}}_{\frac{L^2}{\delta^2}=Re_L}}_{1}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}$$

$$\tilde{v}_x\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = Re_L\frac{\partial \tilde{P}}{\partial \tilde{y}} + \frac{1}{Re_L}\frac{\partial^2 v_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}$$

this is big!, $O(1/\varepsilon)$. Divide everyone by Re_L

$$\frac{1}{Re_L}\left(\tilde{v}_x\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y\frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right) = \frac{\partial \tilde{P}}{\partial \tilde{y}} + \frac{1}{Re_L^2}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{1}{Re_L}\frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}$$

$$\lim_{Re_L\to\infty} \Rightarrow \boxed{\frac{\partial \tilde{P}}{\partial \tilde{y}} = 0}, \quad \boxed{\frac{\partial P}{\partial y} = 0}$$

Supplemental Notes : Helmholtz decomposition

see also
BSL §4.2, §4.3

* Helmholtz theorem says :

$$\underline{v} = -\underline{\nabla} A + \underline{\nabla} \times \underline{B} \qquad (1)$$

(*) see wikipedia for proofs.

for any sufficiently smooth, rapidly decaying vector field.
(faster than $1/r$)

$\underline{\nabla} A$: a curl-free component of $\underline{v}$ (irrotational)

$\underline{\nabla} \times \underline{B}$: a divergence-free component of $\underline{v}$ (incompressible)

(*) recall : $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = 0$, $\underline{\nabla} \times (\nabla A) = 0$ } fundamental properties of vector calculus.

* <u>Important</u> : A key fact to realize about the Helmholtz decomposition is that it is not unique. For a given $\underline{v}$, we can find different combinations of A & $\underline{B}$ that satisfy (1).

<u>Example</u> : $\underline{v} = \underline{e}_x = [1, 0, 0]$

$$\underline{\nabla} \cdot \underline{v} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(0) = 0$$

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} e_x & e_y & e_z \\ \partial x & \partial y & \partial z \\ 1 & 0 & 0 \end{vmatrix} = [0, 0, 0]$$

(a) $A = -x$, $\underline{B} = [0, 0, 0] \Rightarrow \nabla A = \underline{e}_x$, $\underline{\nabla} \times \underline{B} = 0$

(b) $A = 0$, $\underline{B} = [0, 0, y] \Rightarrow \underline{\nabla} A = \underline{0}$

$$\underline{\nabla} \times \underline{B} = \begin{vmatrix} e_x & e_y & e_z \\ \partial x & \partial y & \partial z \\ 0 & 0 & y \end{vmatrix} = \underline{e}_x \frac{\partial y}{\partial y} = \underline{e}_x$$

* In other words, it is possible to find non-trivial choices of A & $\underline{B}$ for a vector field, even if it is irrotational & curl-free.

* So $\underline{\nabla} A$ & $\underline{\nabla} \times \underline{B}$ are "a" curl-free & divergence-free component of $\underline{v}$, not "the" " " & " " components. (Respectively)

* However a choice of one (A or $\underline{B}$) fixes the other.

* In fluid mechanics, we often deal with vector fields that are both curl-free (irrotational) & divergence free (incompressible). Note that a lack of uniqueness means

$$\text{curl-free} \not\Rightarrow \underline{\nabla} \times \underline{B} = 0 \quad , \quad \underline{v} = \underline{\nabla} A$$

$$\text{div-free} \not\Rightarrow \underline{\nabla} A = 0 \quad , \quad \underline{v} = \underline{\nabla} \times \underline{B}$$

(This is the converse of the Helmholtz decomposition. Logic doesn't work when not unique!)

* However, we choose to define the following:

(a) $\underline{v} = -\underline{\nabla}\phi$, $\underline{B} = 0$ velocity potential

(b) $\underline{v} = \underline{\nabla} \times \underline{\Psi}$, $A = 0$ stream function

Both are valid choices. (a) is useful for Irrotational flow. (b) is useful for 2D / axisymmetric flows.

Supplemental Notes: Blasius Equation Derivation

Supplemental Notes - Blasius' Equation

$$\eta = y/\delta(x) \qquad \delta(x) = \left(\frac{2\nu x}{u}\right)^{1/2}$$

$$\frac{v_x}{u} = \frac{df}{d\eta} = f' \quad \leftarrow \text{let } f' = \frac{df}{d\eta}, \quad f'' = \frac{d^2 f}{d\eta^2}$$

$$f''' = \frac{d^3 f}{d\eta^3}$$

substitute into:

$$\underset{(1)}{v_x} \underset{(2)}{\frac{\partial v_x}{\partial x}} + \underset{(4)}{v_y} \underset{(3)}{\frac{\partial v_x}{\partial y}} = \nu \underset{(5)}{\frac{\partial^2 v_x}{\partial y^2}}$$

(1) $v_x = u \dfrac{df}{d\eta} = \underline{u f'}$

chain rule: $\partial/\partial x$

(2) $$\frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x}\left(u \frac{df}{d\eta}\right) = u \frac{\partial \eta}{\partial x}\frac{d}{d\eta}\frac{df}{d\eta} = u \frac{\partial \eta}{\partial x}\frac{d^2 f}{d\eta^2}$$

$$= u \frac{\partial \eta}{\partial x} f''$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{\delta(x)}\right) = -y\,\delta^{-2}\frac{\partial \delta}{\partial x} = -\eta\,\delta^{-1}\frac{\partial \delta}{\partial x}$$

$$\frac{\partial \delta}{\partial x} = \frac{1}{2}\left(\frac{2\nu x}{u}\right)^{-1/2} \cdot \frac{2\nu}{u} = \frac{\nu}{u}\delta^{-1}$$

$$\frac{\partial \eta}{\partial x} = -\eta\,\delta^{-1} \cdot \frac{\nu}{u}\delta^{-1} = -\eta \nu/u\ \delta^{-2}$$

$$= u\left(-\frac{\eta\nu}{u}\delta^{-2}\right) f'' = \underline{-\eta\nu \frac{f''}{\delta^2}}$$

(3) $$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left(u \frac{df}{d\eta}\right) = u \frac{\partial \eta}{\partial y}\frac{d}{d\eta}\frac{df}{d\eta}$$

chain rule: $\partial/\partial y$

$$= u \frac{\partial \eta}{\partial y}\frac{d^2 f}{d\eta^2} = u \frac{\partial \eta}{\partial y} f''$$

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left(y/\delta\right) = 1/\delta$$

$$= \underline{\frac{u f''}{\delta}}$$

(4) $v_y = -\int \frac{\partial v_x}{\partial x}\, dy = +\int \frac{\eta\nu}{u}\frac{f''}{\delta^2}\underbrace{\delta\, d\eta}$

$\frac{d\eta}{dy} = 1/\delta \Rightarrow \quad dy = \delta\, d\eta$

$= +\int \eta\nu \frac{f''}{\delta}\, d\eta$

integration by parts: $\int u\, dv = uv - \int v\, du$

$u = \eta \cdot \nu/\delta \qquad dv = f''\, d\eta = \frac{d^2 f}{d\eta^2}\, d\eta$

$du = \frac{\nu}{\delta}\, d\eta \qquad v = \frac{df}{d\eta} = f'$

$= \left[\frac{\eta\nu}{\delta} f' - \int \frac{\nu}{\delta} f'\, d\eta\right]$

$= \frac{\eta\nu}{\delta} f' - \frac{\nu}{\delta}\int \frac{df}{d\eta}\, d\eta = \frac{\eta\nu}{\delta} f' - \frac{\nu}{\delta} f$

$= \underline{\frac{\nu}{\delta}\left[\eta f' - f\right]}$

(5) $\frac{d^2 v_x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}\right) = \underset{\uparrow \frac{1}{\delta}}{\frac{\partial \eta}{\partial y}}\frac{d}{d\eta}\underset{\uparrow \frac{uf''}{\delta}}{\left(\frac{\partial v_x}{\partial y}\right)} = \frac{1}{\delta}\frac{d}{d\eta}\left(\frac{uf''}{\delta}\right)$

$= \frac{u}{\delta^2}\frac{d}{d\eta}\left(\frac{d^2 f}{d\eta^2}\right) = \underline{\frac{u}{\delta^2} f'''}$

* combine all together:

$\overset{(1)}{u f'}\overset{(2)}{\left(-\eta\nu\frac{f''}{\delta^2}\right)} + \overset{(4)}{\frac{\nu}{\delta}\left[\eta f' - f\right]}\overset{(3)}{\left(\frac{uf''}{\delta}\right)} = \nu \cdot \overset{(5)}{\frac{u}{\delta^2} f'''}$

$\cancel{-uf'f''\frac{\eta\nu}{\delta^2} - u\frac{\eta\nu}{\delta^2}f'f''} - u\frac{\nu}{\delta^2} ff'' = \nu\frac{u}{\delta^2} f'''$

$-\frac{\cancel{u\nu}}{\cancel{\delta^2}} ff'' = \frac{+\cancel{u\nu}}{\cancel{\delta^2}} f''' \Rightarrow \quad -ff'' = f'''$

$$\boxed{f''' + ff'' = 0}$$