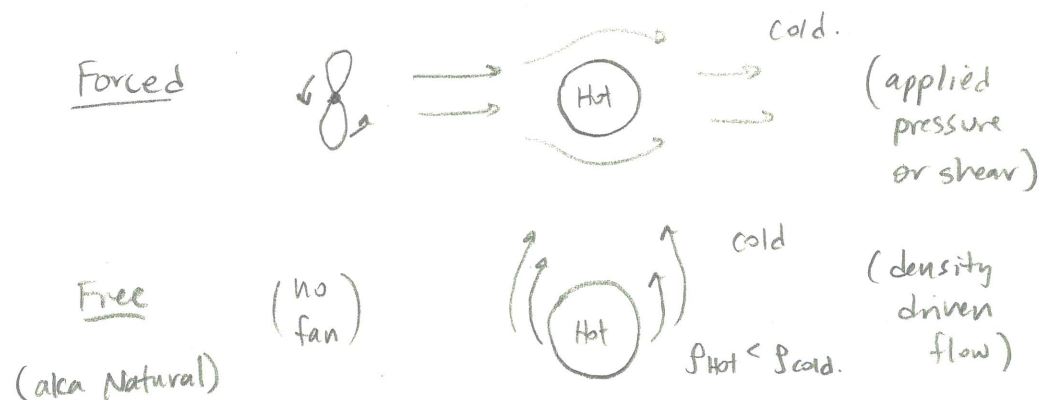


Lecture 22: Forced Convection - Internal Laminar Flows

I. Types of convective Heat and Mass Transfer

* Now that we have done heat/mass transport and fluid flow, we can put them together and do convective heat/mass transport problems

* There are two different types of convection problems: forced convection & free convection



* In forced convection, the velocity field is largely unaffected by the heat/mass transfer, we can solve for $\underline{u}$ first using a momentum balance. Then we can solve the heat/mass transfer problem.

(In the end, we will get $Nu = Nu(Re, Pr)$)

* In free convection, the velocity field is created by the heat/mass transfer. So, we have to solve the energy (T) and momentum ($\underline{u}$) equations simultaneously

(In the end, we will get $Nu = Nu(Gr, Pr)$)

* Convection problems are also distinguished by the geometry of the flow.

Internal Flows
(Confined Flows)



prototypical
example:
pipe flow

External Flows
(Unconfined Flows)



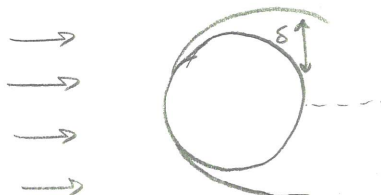
prototypical
example:
flow over
a pipe.

* In confined flows, we have walls that enclose our flow. Consequently, at steady state, any boundary layers eventually grow to span the domain. Such a flow is considered fully developed and can often be unidirectional.



entrance region where BL grow.

* In unconfined flows, the walls are internal and we have some boundary condition at infinity. Consequently, at steady state we have finite boundary layers that we must consider (for flow at high Re). Such flows are almost always at least bidirectional.



- * Finally, in both forced and free convection and for both confined and unconfined flows, we may have turbulence. The latter introduces serious complications to our analysis of these processes. For now, we will consider so-called "laminar" flows.

II. "Practical" Heat/Mass Transfer and Dimensional Analysis

- * Let us consider forced-convection heat/mass transfer for a confined laminar flow.

Energy Equation $\frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$

Species Mass Equation $\frac{\partial c_i}{\partial t} + \underline{v} \cdot \underline{\nabla} c_i = D_i \nabla^2 c_i$

↗ In the past, we let $\underline{v}$ be zero.
No longer!

- * Restrict ourselves to steady problems.

$$\frac{\partial T}{\partial t} = 0 \Rightarrow \underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$$

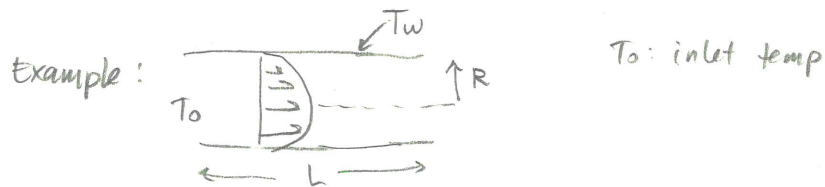
$$\frac{\partial c_i}{\partial t} = 0 \Rightarrow \underline{v} \cdot \underline{\nabla} c_i = D_i \nabla^2 c_i$$

- * How get $\underline{v}$? By solving the momentum equation (Navier Stokes)

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) = -\underline{\nabla} p + \mu \nabla^2 \underline{v}$$

$$\frac{\partial \underline{v}}{\partial t} = 0 \Rightarrow \underline{v} \cdot \underline{\nabla} \underline{v} = -\frac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{v}$$

* Let us Non-dimensionalize this problem. We will focus on heat transfer for now. Note that the equations for H.T and M.T. (dilute) are the same!



$$\tilde{\nabla} = R \nabla \quad \theta = \frac{T - T_w}{T_0 - T_w} \quad \tilde{u} = \frac{u}{u}$$

$$u \tilde{u} \cdot \frac{\Delta T}{R} \tilde{\nabla} \theta = \alpha \frac{\Delta T}{R^2} \tilde{\nabla}^2 \theta$$

$$\frac{uR}{\alpha} \tilde{u} \cdot \tilde{\nabla} \theta = \tilde{\nabla}^2 \theta$$

$$Pe = \frac{uR}{\alpha} = \frac{u}{\alpha/R} \quad \begin{array}{l} \text{convective} \\ \text{velocity} \\ \hline \text{conductive} \\ \text{velocity} \end{array}$$

$$\boxed{Pe \tilde{u} \cdot \tilde{\nabla} \theta = \tilde{\nabla}^2 \theta}$$

Also recall that: (when $\tilde{p} = P/\rho u^2$)

$$\frac{u^2}{R} \tilde{u} \cdot \tilde{\nabla} \tilde{u} = -\frac{1}{\rho} \frac{\rho u^2}{R} \tilde{\nabla} \tilde{p} + \frac{\mu}{\rho} \frac{u}{R^2} \tilde{\nabla}^2 \tilde{u}$$

$$\tilde{u} \cdot \tilde{\nabla} \tilde{u} = -\tilde{\nabla} \tilde{p} + \frac{\mu}{\rho R u} \tilde{\nabla}^2 \tilde{u}$$

$$\boxed{\tilde{u} \cdot \tilde{\nabla} \tilde{u} = -\tilde{\nabla} \tilde{p} + \frac{1}{Re} \tilde{\nabla}^2 \tilde{u}}$$

$$Re = \frac{\rho u R}{\mu}$$

* From this we conclude that

$$\tilde{u} = \tilde{u}(\tilde{r}, Re, \text{geometric ratios}) \quad \begin{array}{l} \text{ratios of length} \\ \text{scales. E.g. } \frac{R}{L} \end{array}$$

$$\theta = \theta(\tilde{r}, Re, Pe, \text{geometric ratios})$$

- Sometimes the latter is written as

$$\theta = \theta(\tilde{r}, Re, Pr, \text{geometric ratios}) \quad (\text{HT})$$

$$\theta = \theta(\tilde{r}, Re, Sc, \text{geometric ratios}) \quad (\text{MT})$$

$$Pr = \frac{\nu}{\alpha} \quad Sc = \frac{\nu}{D_i} \quad (\text{ratios of "diffusivities"})$$

$$(\text{HT}) \quad Pe = Re Pr = \frac{\rho U R}{\mu} \cdot \frac{\nu}{\alpha} = \frac{UR}{\alpha} \checkmark$$

$$(\text{MT}) \quad Pe = Re Sc = \frac{\rho U R}{\mu} \cdot \frac{\nu}{D_i} = \frac{UR}{D_i} \checkmark$$

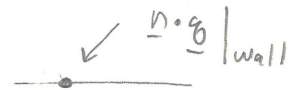
- * In "practical"/engineering contexts, we want to know the total heat transfer (mass transfer).

$$Q = \int \underline{n} \cdot \underline{q} \big|_{\text{wall}} dA \quad \text{or} \quad \dot{m}_i = \int \underline{n} \cdot \underline{N}_i \big|_{\text{wall}} dA$$

- * Focusing on H.T. (just to be succinct), we want

to know $\underline{q}$, the heat flux.

$$\underline{n} \cdot \underline{q} \big|_{\text{wall}} = -k (\underline{n} \cdot \nabla T) \big|_{\text{wall}} = -k \frac{\partial T}{\partial n} \big|_{\text{wall}}$$



- * Conventionally, engineers prefer to write heat fluxes using a heat transfer coefficient

$$q'' = h (T_b - T_w) \quad (q'' \text{ in Incropera})$$

$\uparrow$ bulk or mean temperature $\nwarrow$ wall temperature

bulk or
mean temperature
- velocity averaged
- "flow rate"
averaged

$$T_b = \frac{\int v_z T dA}{\int v_z dA}$$

- think of this as a definition of h . A way to conveniently express heat fluxes.

$$h = \frac{-k \partial T / \partial n |_{\text{wall}}}{T_b - T_w}$$

- * we would prefer to write this in a dimensionless form.

$$h = \frac{-k \cdot \frac{\Delta T}{R} \cdot \frac{\partial \theta}{\partial \tilde{n}} |_w}{\Delta T (\theta_b - \theta_w)} \Rightarrow \frac{hR}{k} = - \frac{\partial \theta}{\partial \tilde{n}} |_w$$

$$Nu = \frac{hR}{k} \quad \text{Nusselt number.}$$

(dimensionless HT coefficient)

$$Nu = \frac{-\partial \theta / \partial \tilde{n} |_w}{\theta_b - \theta_w}$$

- * If we can get Nu for a given problem, then we can use it for engineering calculations.

- * From our dimensional analysis, we conclude that

$$Nu = Nu(\tilde{r}, Re, Pr, \text{geometric ratios}) \quad \text{just like } \theta$$

- if we average over the domain:

$$\bar{Nu} = \bar{Nu}(Re, Pr, \text{geometric ratios})$$

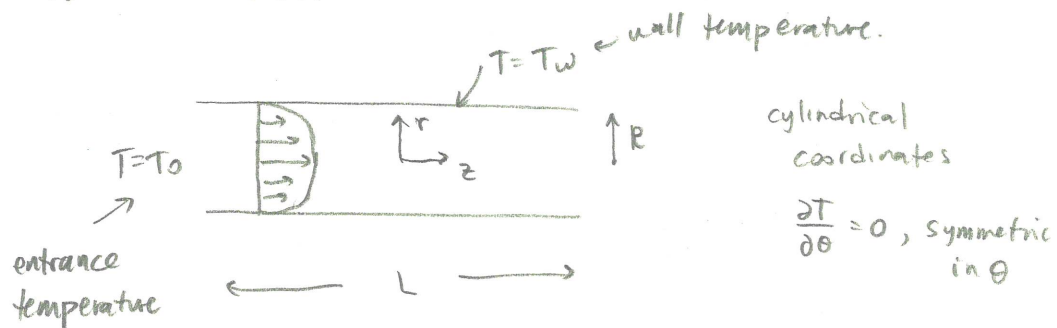
- * This follows similarly for mass transfer.

$$Sh = \frac{k_i R}{D_i}, \quad k_i: \text{mass transfer coefficient for species } i$$

$$Sh = Sh(\tilde{r}, Re, Sc_i, \text{geometric ratios})$$

III. The Graetz Problem (setup)

- * The Graetz problem is a classical example of heat transfer in confined flow. It assumes we have laminar flow in a tube.



$$v_z = 2U \left(1 - \frac{r^2}{R^2}\right)$$

U : mean velocity
($v_r = v_\theta = 0$)

- * we want to solve for $T(r, z)$ and then use this to get Nu .

* Energy equation: $\underline{v} \cdot \underline{\nabla} T = \alpha \nabla^2 T$

$$\underbrace{v_r \frac{\partial T}{\partial r}}_{v_r=0} + \underbrace{\frac{v_\theta}{r} \frac{\partial T}{\partial \theta}}_{v_\theta=0} + v_z \frac{\partial T}{\partial z} = \alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

$\frac{\partial T}{\partial \theta} = 0$

$$v_z \frac{\partial T}{\partial z} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\boxed{2U \left(1 - \frac{r^2}{R^2}\right) \frac{\partial T}{\partial z} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \alpha \frac{\partial^2 T}{\partial z^2}} \quad \leftarrow \text{PDE for } T(r, z)$$

Boundary
Conditions

$$\boxed{T(r, 0) = T_0}$$

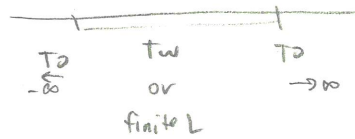
* other z B.C.
not important.

$$\boxed{T(R, z) = T_w}$$

$$\boxed{\frac{\partial T}{\partial r} \Big|_{r=0} = 0} \quad (\text{symmetry})$$

* Aside: This is the "extended" Graetz problem with axial convection term. I see people solve it where:

$$\propto \frac{\partial^2 T}{\partial z^2}$$



$$T(r, 0) = T_0 \text{ and } T(r, L) = T_0$$

$$\text{or } T(r, -\infty) = T_0 \text{ and } T(r, \infty) = T_0$$

$$\text{or possibly } T(r, 0) = T_0 \text{ and } \left. \frac{\partial T}{\partial z} \right|_{z=\infty} = 0$$

These B.C.'s will likely all give the same answer,

as long as L is big enough. $T = T(z)$ should be

flat in the middle of the tube (axially)

* Let's non-dimensionalize:

$$\eta = \frac{r}{R}, \quad \theta = \frac{T - T_0}{T_w - T_0}, \quad \text{figure out } z \text{ in a minute.}$$

↑

$$\Delta T = T_w - T_0$$

(O.O.M. analysis)

$$\xi = z/l \leftarrow \text{what } l?$$

$$2u(1-\eta^2) \cdot \frac{\Delta T}{l} \frac{\partial \theta}{\partial \xi} = \frac{\alpha \Delta T}{R^2} \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + \frac{\alpha \Delta T}{l^2} \frac{\partial^2 \theta}{\partial \xi^2}$$

$$(1-\eta^2) \frac{\partial \theta}{\partial \xi} = \frac{\alpha l}{2uR^2} \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + \frac{\alpha}{2ul} \frac{\partial^2 \theta}{\partial \xi^2}$$

$$u + Pe = \frac{2uR}{\alpha} \qquad \frac{\alpha}{2uR} \cdot \frac{R}{l}$$

$$(1-\eta^2) \frac{\partial \theta}{\partial \xi} = \frac{1}{Pe} \frac{l}{R} \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + \frac{1}{Pe} \frac{R}{l} \frac{\partial^2 \theta}{\partial \xi^2}$$

↑

↑

2 groups to
deal with

• some choices: 1) $l = L \gg R$

usually $Pe \gg 1$
Flow in a pipe
is pretty fast.

$$\frac{1}{Pe} \frac{L}{R}$$

$$\frac{1}{Pe} \frac{R}{L}$$

really small

• we need to
know how
 $L \leftrightarrow Pe$
or else
can't tell
the scale.

2) $l = R$

"What measuring stick
should we choose?"

$$\frac{1}{Pe} \cdot \frac{R}{R}$$

$$\frac{1}{Pe} \cdot \frac{R}{R}$$

• unrealistically
short pipe

• Like a semi-infinite
problem for z .

what scale for z ?

3) $l = Pe R$ (will have $l \gg R$)

$$\frac{1}{Pe} \cdot \frac{Pe R}{R} = 1$$

$$\frac{1}{Pe} \cdot \frac{R}{Pe R} = \frac{1}{Pe^2}$$

• Why do I have
a D.O.F. to
pick z ?

Because at
high Pe , I don't
have another B.C.
because I lose the
axial diffusion term.
Just like semi-
infinite diffusion.

A special "L"
where the terms
balance. Need $\frac{dT}{dz}$
to be small enough for
convection to match radial
conduction.

↑
This gives us the scale for z , where
radial conduction and convection
are balanced. Just as
much heat is being conducted
in as flow is carrying away.

* using $\zeta = z / R Pe$ gives

$$(1 - \eta^2) \frac{\partial \theta}{\partial \zeta} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) + \frac{1}{Pe^2} \frac{\partial^2 \theta}{\partial \zeta^2}$$

* we typically care about the case when $Pe \gg 1$.

In this case, axial (down the pipe) diffusion is
not important.

$$(1 - \eta^2) \frac{\partial \theta}{\partial \zeta} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right)$$

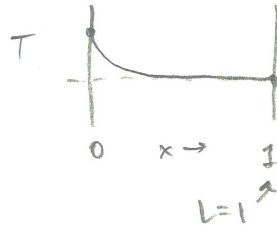
$$\theta(\eta, \zeta=0) = 0, \quad \theta(\eta=1, \zeta) = 1, \quad \frac{\partial \theta}{\partial \eta}(\eta=0, \zeta) = 0$$

Dimensionless Graetz problem.

IV. Solution to the Graetz Problem

* There are two useful solutions to the Graetz problem.

A useful analogy here is the transient diffusion problem:



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \begin{aligned} T(x, 0) &= 0 \\ T(0, t) &= 1 \\ T(1, t) &= 0 \end{aligned}$$

$$T(x, t) = \underbrace{1-x}_{T_{ss}} - 2 \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n\pi} e^{-n^2\pi^2 t}$$

But at early times
($t \ll L^2/D^2$)

the similarity solution: $T = 1 - \frac{2}{\sqrt{\pi}} \int_0^M e^{-s^2} ds, \quad M = \frac{x}{\sqrt{4\alpha t}}$

gives an accurate solution.

(Note: at early times, the similarity solution is preferred, because the series solution converges slowly for small t .)

* Similarly for the Graetz problem, we have a series solution & a similarity solution. Note the analogy between the S variable and time in the transient diffusion problem. $\hookrightarrow$ (this is space in z -axis, with diffusion now being radial for Graetz).

• so the small- S solution for the Graetz problem is an "entrance region" rather than "early time."

A. Entrance Region Solution for Graetz problem (similarity)

$$(1-\eta^2) \frac{\partial \theta}{\partial \zeta} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right)$$



let $\chi = 1-\eta$ so χ is small when $\zeta \ll 1 \Rightarrow \chi \ll 1$

$$[1 - (1-\chi)^2] \frac{\partial \theta}{\partial \zeta} = \frac{1}{(1-\chi)} \frac{\partial}{\partial (1-\chi)} \left[(1-\chi) \frac{\partial \theta}{\partial (1-\chi)} \right]$$

$$[\chi - \chi + 2\chi - \chi^2] \frac{\partial \theta}{\partial \zeta} = \frac{1}{1-\chi} \frac{\partial}{\partial \chi} \left[(1-\chi) \frac{\partial \theta}{\partial \chi} \right]$$

$\chi^2 \ll \chi \leftarrow \textcircled{*}$ $\rightarrow 1-\chi \approx 1$ (essentially low curvature)

(linear velocity profile)

$$\boxed{2\chi \frac{\partial \theta}{\partial \zeta} = \frac{\partial^2 \theta}{\partial \chi^2}}$$

$\textcircled{*}$ Is called the L  v  que approximation.

$$\boxed{\theta(\chi, 0) = 0}, \quad \boxed{\theta(0, \zeta) = 1}, \quad \boxed{\theta(\infty, \zeta) = 0}$$

replaces symmetry.

Says $T = T_0$ in center.

* Now, suppose that

$$\theta = \theta(s) \quad \text{where} \quad s = \frac{\chi}{g(\zeta)} \quad \zeta = \frac{z}{R Pe} \quad \chi = 1 - \frac{r}{R}$$

* Convert the PDE to similarity variables.

$$\frac{\partial \theta}{\partial \zeta} = \frac{\partial s}{\partial \zeta} \frac{\partial \theta}{\partial s}$$

$$\frac{\partial s}{\partial \zeta} = \frac{\partial}{\partial \zeta} \left(\frac{\chi}{g} \right) = \chi (-g^{-2}) g'$$

$$= -\frac{\chi g'}{g^2} \frac{\partial \theta}{\partial s}$$

$$= -\frac{\chi g'}{g^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial \theta}{\partial s} \cdot \frac{\partial}{\partial x} \left(\frac{x}{g} \right) = \frac{1}{g} \frac{\partial \theta}{\partial s}$$

$$\begin{aligned} \frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{1}{g} \frac{\partial \theta}{\partial s} \right) = \frac{1}{g} \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial s} \right) \\ &= \frac{1}{g} \frac{\partial s}{\partial x} \frac{\partial}{\partial s} \left(\frac{\partial \theta}{\partial s} \right) = \frac{1}{g} \cdot \frac{1}{g} \frac{\partial^2 \theta}{\partial s^2} = \frac{1}{g^2} \frac{\partial^2 \theta}{\partial s^2} \end{aligned}$$

* Put it together

$$2x \left(-\frac{xg'}{g^2} \right) \frac{\partial \theta}{\partial s} = \frac{1}{g^2} \frac{\partial^2 \theta}{\partial s^2}$$

$$-2 \frac{x^2}{g^2} g' \frac{\partial \theta}{\partial s} = \frac{1}{g^2} \frac{\partial^2 \theta}{\partial s^2} \quad s = \frac{x}{g}$$

$$-2s^2 g' \frac{\partial \theta}{\partial s} = \frac{1}{g^2} \frac{\partial^2 \theta}{\partial s^2}$$

$$\boxed{\frac{\partial^2 \theta}{\partial s^2} + 2s^2 g' g^2 \frac{\partial \theta}{\partial s} = 0}$$

* Solve for g:

$$g^2 \frac{dg}{ds} = c$$

$$g^2 \frac{dg}{ds} = \frac{d}{ds} (g^3) \cdot \frac{1}{3}$$

$$\int \frac{1}{3} d(g^3) = \int c \, ds \Rightarrow \frac{1}{3} g^3 = cs + c_1$$

$$\text{let } g(0) = 0 \Rightarrow c_1 = 0$$

$$g^3 = 3s \cdot c \quad g = (3s \cdot c)^{1/3}$$

$$\text{Deen picks } c = \frac{3}{2} \Rightarrow \boxed{g = \left(\frac{9}{2} s \right)^{1/3}}$$

$$g^2 g' = \frac{3}{2}$$

plug into above

$$\boxed{s = \frac{x}{g} = \left(\frac{2}{9} \right)^{1/3} \frac{x}{s^{1/3}}}$$

* now fix BCs for similarity problem

$$\theta(x,0) = 0 \Rightarrow s = \left(\frac{2}{9}\right)^{1/3} \cdot \frac{x}{0^{1/3}} = \infty$$

$$\boxed{\theta(\infty) = 0}$$

$$\theta(0,s) = 1 \Rightarrow s = \left(\frac{2}{9}\right)^{1/3} \cdot \frac{0}{s^{1/3}} = 0$$

$$\boxed{\theta(0) = 1}$$

$$\theta(\infty,s) = 0 \Rightarrow s = \left(\frac{2}{9}\right)^{1/3} \cdot \frac{\infty}{s^{1/3}} = \infty$$

* Now solve similarity ODE

$$\frac{d^2\theta}{ds^2} + 3s^2 \frac{d\theta}{ds} = 0 \quad \theta(0)=1, \quad \theta(\infty)=0$$

$$\text{let } u = \frac{d\theta}{ds} \quad \frac{du}{ds} = \frac{d^2\theta}{ds^2}$$

$$\frac{du}{ds} + 3s^2 u = 0 \Rightarrow \frac{du}{ds} = -3s^2 u$$

$$\Rightarrow \frac{1}{u} \frac{du}{ds} = -3s^2 \Rightarrow \ln u = -s^3 + C_1$$

$$u = \exp(-s^3) \cdot C_1'$$

$$\frac{d\theta}{ds} = C_1' \exp(-s^3)$$

$$\theta = \int_0^s C_1' \exp(-t^3) dt + C_2$$

• Apply B.C's: $\theta(0)=1$
 $\theta(\infty)=0$

$$\theta(0) = \int_0^0 c_1' \exp(-t^3) + c_2 = 1 \Rightarrow c_2 = 1$$

$$\theta(\infty) = \int_0^{\infty} c_1' \exp(-t^3) + 1 = 0$$

$$c_1' = \frac{-1}{\int_0^{\infty} \exp(-t^3) dt}$$

$$\theta(s) = \frac{-\int_0^s \exp(-t^3) dt}{\int_0^{\infty} \exp(-t^3) dt} + 1$$

← can solve via a trick
with gamma functions
and u-substitution

$$I = \int_0^{\infty} e^{-t^3} dt$$

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \quad (\text{gamma func})$$

$$\text{let } x = t^3 \Rightarrow t = x^{1/3}$$

$$dt = \frac{1}{3} x^{-2/3} dx$$

$$= \int_0^{\infty} e^{-x} \cdot \frac{1}{3} x^{-2/3} dx$$

$$= \frac{1}{3} \int_0^{\infty} x^{-2/3} e^{-x} dx = \frac{1}{3} \Gamma(1/3)$$

$$\begin{array}{c} \uparrow \\ x^{1/3-1} \Rightarrow z = 1/3 \end{array}$$

$$\text{So, } \theta(s) = \frac{-\int_0^s \exp(-t^3) dt}{\frac{1}{3} \Gamma(1/3)} + 1 = \frac{-\int_0^s e^{-t^3} dt + \int_0^{\infty} e^{-t^3} dt}{\frac{1}{3} \Gamma(1/3)}$$

$$\boxed{\theta(s) = \frac{3}{\Gamma(1/3)} \int_s^{\infty} \exp(-t^3) dt}$$

$$\boxed{s = \left(\frac{2}{9}\right)^{1/3} \frac{x}{5^{1/3}}}$$

* Now, to find the Nusselt number:

$$Nu = \frac{2hR}{k} = \frac{-2 \frac{\partial \theta}{\partial x} \big|_{x=0}}{\theta_w - \theta_b} = -2 \frac{\partial \theta}{\partial x} \big|_{x=0}$$

$\uparrow$ for tube diameter $\uparrow$ $\theta_w = 1$ $\uparrow$ $\theta_b = 0$

$$\frac{d\theta}{dx} = \frac{d\theta}{ds} \frac{ds}{dx} = \frac{1}{g} \frac{d\theta}{ds} \quad s = x/g$$

(from above) $\frac{d\theta}{ds} = c_1' \exp(-s^3) \quad c_1' = \frac{-1}{\int_0^\infty \exp(-t^3)} = \frac{-3}{\Gamma(4/3)}$

$$\frac{d\theta}{ds} = \frac{-3}{\Gamma(4/3)} \exp(-s^3)$$

$$\frac{d\theta}{ds} \big|_{x=0} = \frac{d\theta}{ds} \big|_{s=0} = \frac{-3}{\Gamma(4/3)}$$

$$Nu = -2 \cdot \frac{1}{g} \cdot \frac{(-3)}{\Gamma(4/3)} = \frac{6}{\Gamma(4/3) g(s)}$$

$$Nu = \frac{6}{\Gamma(4/3)} \frac{1}{g(s)} \quad g(s) = \left(\frac{9}{2} s\right)^{1/3} \quad s = \frac{x}{R Pe}$$

$$Nu = \underbrace{\frac{6}{\Gamma(4/3)} \left(\frac{2}{9}\right)^{1/3}}_{1.3566} \cdot \left(\frac{R}{2}\right)^{1/3} (Pe)^{1/3}$$

B. Fully Developed Solution for Graetz Problem (FFT)

$$\frac{\partial \theta}{\partial \xi} = \frac{1}{\eta(1-\eta^2)} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right)$$

$$\text{let } \theta = \frac{T - T_w}{T_o - T_w}$$

it swaps the BCs

These two "swap" $\rightarrow \theta(\eta, \xi=0) = 1 \quad \leftarrow T(r, 0) = T_o$

$\rightarrow \theta(\eta=1, \xi) = 0 \quad \leftarrow T(R, z) = T_w$

$\frac{\partial \theta}{\partial \eta}(\eta=0, \xi) = 0 \quad \leftarrow \frac{\partial T}{\partial r}(0, z) = 0$

Eigenfunction:

$$\frac{1}{\eta(1-\eta^2)} \frac{d}{d\eta} \left(\eta \frac{d\Phi}{d\eta} \right) = -\lambda^2 \Phi$$

is Sturm-Liouville $\Rightarrow$ cylindrical Graetz functions

$$\Phi_n(\eta) = a_n G(\lambda_n \eta)$$

$\nwarrow$ normalization constant.

$$w(\eta) = (1-\eta^2)\eta, \quad p(\eta) = \eta, \quad q(\eta) = 0$$

FFT:

$$\hat{\theta}_n = \int_0^1 \theta \cdot \Phi_n (1-\eta^2) \eta d\eta$$

• Apply to PDE gives:

$$\text{LHS: } \int_0^1 \frac{\partial \theta}{\partial \xi} \Phi_n (1-\eta^2) \eta d\eta$$

$$= \frac{\partial}{\partial \xi} \int_0^1 \theta \Phi_n (1-\eta^2) \eta d\eta = \frac{\partial \hat{\theta}_n}{\partial \xi}$$

$$\text{RHS: } \int_0^1 \underbrace{\frac{1}{\eta(1-\eta^2)} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right)} \cdot \Phi_n(1-\eta^2) \eta d\eta$$

By the eigenvalue problem

this equals $-\lambda_n^2 \theta$

$$-\lambda^2 \int_0^1 \theta \Phi_n(1-\eta^2) \eta d\eta = -\lambda_n^2 \hat{\theta}_n$$

$$\text{so: } \frac{\partial \hat{\theta}}{\partial \xi} = -\lambda_n^2 \hat{\theta}_n \Rightarrow \hat{\theta}_n = \exp(-\lambda_n^2 \xi) \cdot b_n$$

$$b_n = \int_0^1 \underbrace{\theta(\eta, \xi=0)}_1 \Phi_n(1-\eta^2) \eta d\eta \quad (\text{FFT of initial condition})$$

$$b_n = \int_0^1 \Phi_n(1-\eta^2) \eta d\eta$$

* Put it all together:

$$\theta(\eta, \xi) = \sum_{n=1}^{\infty} b_n \exp(-\lambda_n^2 \xi) \cdot a_n G(\lambda_n \eta)$$

$$\boxed{\theta(\eta, \xi) = \sum_{n=1}^{\infty} c_n \exp(-\lambda_n^2 \xi) G(\lambda_n \eta)}, \quad c_n = a_n b_n$$

* The trick to this one is getting the Graetz functions and the eigenvalues.

* Apparently, Graetz functions are more easily found via "confluent hypergeometric functions"

$$G(\lambda\eta) = e^{-\lambda\eta^2/2} M\left(\frac{1}{2} - \frac{\lambda}{4}, 1, \lambda\eta^2\right)$$

with eigenvalues given by zeros:

$$M\left(\frac{1}{2} - \frac{\lambda}{4}, 1, \lambda\right) = 0$$

* Now to get Nu :

$$Nu(\xi) = \frac{2hR}{k} = - \frac{2(\partial\theta/\partial\eta)|_{\eta=1}}{\theta_b(\xi) - \theta_w} \leftarrow \theta_w = \theta(\eta=1) = 0$$

$$\frac{\partial\theta}{\partial\eta} = \frac{\partial}{\partial\eta} \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} G(\lambda_n \eta) \quad (\text{go term by term})$$

$$= \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \frac{\partial G}{\partial\eta}(\lambda_n \eta) \cdot \lambda_n$$

$$\left. \frac{\partial\theta}{\partial\eta} \right|_{\eta=1} = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \frac{\partial G}{\partial\eta}(\lambda_n) \cdot \lambda_n \quad \left. \vphantom{\sum_{n=1}^{\infty}} \right\} \eta=1$$

• Need to get θ_b :

$$T_b = \frac{\int T v_z dA}{\int v_z dA}$$

$$T = \Delta T \theta$$

$$v_z = 2u(1 - r^2/R^2)$$

$$dA = 2\pi r dr$$

$$r = R\eta$$

$$T_b = \frac{\int \Delta T \theta \cdot 2u(1-\eta^2) \cdot 2\pi r dr}{\int 2u(1-\eta^2) \cdot 2\pi r dr}$$

$$\frac{T_b}{\Delta T} = \frac{\int_0^1 \theta(1-\eta^2) \eta d\eta}{\int_0^1 (1-\eta^2) \eta d\eta}$$

→ do this one: $\int_0^1 (\eta - \eta^3) d\eta$

$$\theta_b = 4 \int_0^1 \theta(1-\eta^2) \eta d\eta$$

$$= \left. \frac{\eta^2}{2} - \frac{\eta^4}{4} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

• Using the Graetz PDE:

$$(1-\eta^2) \frac{d\theta}{d\xi} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right)$$

↓

$$\frac{\partial}{\partial \xi} \int_0^1 \underbrace{\theta \cdot \eta(1-\eta^2)}_{\theta_b/4} d\eta = \int_{\eta=0}^{\eta=1} d\left(\eta \frac{\partial \theta}{\partial \eta}\right)$$

$$\frac{1}{4} \frac{\partial \theta_b}{\partial \xi} = \left(\eta \frac{\partial \theta}{\partial \eta} \right) \Big|_{\eta=1} - \left(\eta \frac{\partial \theta}{\partial \eta} \right) \Big|_{\eta=0}$$

↑
this term
is

$$\eta=1, \frac{\partial \theta}{\partial \eta} \Big|_{\eta=1}$$

this term is zero.

$$\frac{\partial \theta_b}{\partial \xi} = 4 \frac{\partial \theta}{\partial \eta} \Big|_{\eta=1}$$

integral w.r.t. ξ

$$\text{So, } \theta_b = -4 \int_{\xi}^{\infty} \frac{\partial \theta}{\partial \eta}(1, t) dt$$

$T_b \rightarrow T_w$
(B/c $\theta_b \rightarrow 0$
as $\xi \rightarrow \infty$)

Integrate,
term by term

$$\rightarrow \theta_b = -4 \int_{\xi}^{\infty} \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t} \frac{\partial G}{\partial \eta}(\lambda_n) \lambda_n dt$$

$$= -4 \sum_{n=1}^{\infty} C_n \cdot \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2} \frac{\partial G}{\partial \eta}(\lambda_n) \lambda_n$$

negative sign
stays :-

$$\left[\frac{e^{-\lambda^2 \infty}}{-\lambda^2} - \frac{e^{-\lambda^2 \xi}}{-\lambda^2} \right]$$

$$\theta_0 = -4 \sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n} \frac{\partial G}{\partial \eta}(\lambda_n)$$

* Putting it all together

$$Nu = \frac{-2 \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \lambda_n \frac{\partial G}{\partial \eta}(\lambda_n)}{-4 \sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n} \frac{\partial G}{\partial \eta}(\lambda_n)}$$

$$Nu = \frac{1}{2} \frac{\sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 \xi} \lambda_n \frac{\partial G}{\partial \eta}(\lambda_n)}{\sum_{n=1}^{\infty} c_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n} \frac{\partial G}{\partial \eta}(\lambda_n)}$$

keeping only $n=1$ (good when $\xi \rightarrow \infty$, λ_1 dominates because it is smallest).

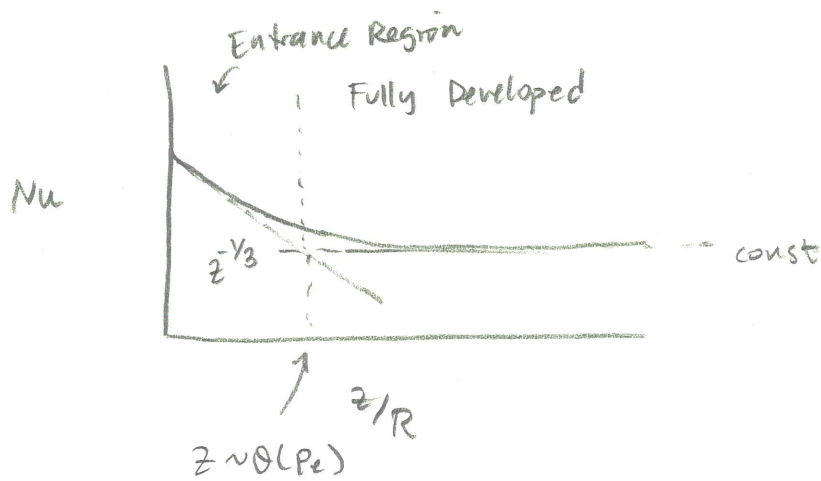
$$Nu(\xi \rightarrow \infty) = \frac{1}{2} \frac{\cancel{c_1} e^{-\lambda_1^2 \xi} \lambda_1 \cdot \frac{\partial G}{\partial \eta}(\lambda_1)}{\cancel{c_1} \frac{e^{-\lambda_1^2 \xi}}{\lambda_1} \cdot \frac{\partial G}{\partial \eta}(\lambda_1)} = \frac{1}{2} \lambda_1^2$$

$$Nu(\infty) = \frac{1}{2} \lambda_1^2$$

From book $\lambda_1 = 2.7044$ so $Nu(\infty) = 3.657$

A constant!

* Conclusion. what does the solution look like?



* Entrance region: radial diffusion is growing

* Fully developed: radial diffusion meets wall and "maxes out"