If $R_n(z)$ is the remainder after n terms then

7.1.24

$$R_n(z)=(-1)^n \frac{1\cdot 3\ldots(2n-1)}{(2z^2)^n}\,\theta,$$

$$\theta=\int_0^\infty e^{-t}\left(1+\frac{t}{z^2}\right)^{-n-\frac{1}{2}}dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

$$|\theta|<1 \qquad \left(|\arg z|<\frac{\pi}{4}\right)$$

For x real, $R_n(x)$ is less in absolute value than the first neglected term and of the same sign.

Rational Approximations[2] ($0\leq x<\infty$)

7.1.25

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3)\,e^{-x^2}+\epsilon(x), \qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 2.5\times 10^{-5}$$

$p=.47047$ $\quad a_1=.34802\ 42$ $\quad a_2=-.09587\ 98$ $\quad a_3=.74785\ 56$

7.1.26

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5)\,e^{-x^2}+\epsilon(x),$$

$$t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 1.5\times 10^{-7}$$

$p=.32759\ 11$ $\quad a_1=.25482\ 9592$

$a_2=-.28449\ 6736$ $\quad a_3=1.42141\ 3741$

$a_4=-1.45315\ 2027$ $\quad a_5=1.06140\ 5429$

7.1.27

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+a_3x^3+a_4x^4]^4}+\epsilon(x)$$

$$|\epsilon(x)|\leq 5\times 10^{-4}$$

$a_1=.278393$ $\quad a_2=.230389$

$a_3=.000972$ $\quad a_4=.078108$

7.1.28

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+\cdots+a_6x^6]^{16}}+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-7}$$

$a_1=.07052\ 30784$ $\quad a_2=.04228\ 20123$

$a_3=.00927\ 05272$ $\quad a_4=.00015\ 20143$

$a_5=.00027\ 65672$ $\quad a_6=.00004\ 30638$

[2] Approximations **7.1.25–7.1.28** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

Infinite Series Approximation for Complex Error Function [7.19]

7.1.29

$$\operatorname{erf}(x+iy)=\operatorname{erf} x+\frac{e^{-x^2}}{2\pi x}\,[(1-\cos 2xy)+i\sin 2xy]$$

$$+\frac{2}{\pi}\,e^{-x^2}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2}\,[f_n(x,y)+ig_n(x,y)]+\epsilon(x,y)$$

where

$$f_n(x,y)=2x-2x\cosh ny\cos 2xy+n\sinh ny\sin 2xy$$

$$g_n(x,y)=2x\cosh ny\sin 2xy+n\sinh ny\cos 2xy$$

$$|\epsilon(x,y)|\approx 10^{-16}|\operatorname{erf}(x+iy)|$$

7.2. Repeated Integrals of the Error Function

Definition

7.2.1

$$\mathrm{i}^n\operatorname{erfc} z=\int_z^\infty \mathrm{i}^{n-1}\operatorname{erfc} t\,dt \qquad (n=0,1,2,\ldots)$$

$$\mathrm{i}^{-1}\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\,e^{-z^2},\ \mathrm{i}^0\operatorname{erfc} z=\operatorname{erfc} z$$

Differential Equation

7.2.2 $$\frac{d^2y}{dz^2}+2z\,\frac{dy}{dz}-2ny=0$$

$$y=A\mathrm{i}^n\operatorname{erfc} z+B\mathrm{i}^n\operatorname{erfc}(-z)$$

(A and B are constants.)

Expression as a Single Integral

7.2.3 $$\mathrm{i}^n\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty\frac{(t-z)^n}{n!}\,e^{-t^2}dt$$

Power Series[3]

7.2.4 $$\mathrm{i}^n\operatorname{erfc} z=\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{2^{n-k}k!\,\Gamma\left(1+\frac{n-k}{2}\right)}$$

Recurrence Relations

7.2.5

$$\mathrm{i}^n\operatorname{erfc} z=-\frac{z}{n}\,\mathrm{i}^{n-1}\operatorname{erfc} z+\frac{1}{2n}\,\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

7.2.6

$$2(n+1)(n+2)\mathrm{i}^{n+2}\operatorname{erfc} z$$

$$=(2n+1+2z^2)\mathrm{i}^n\operatorname{erfc} z-\frac{1}{2}\,\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

[3] The terms in this series corresponding to $k=n+2, n+4, n+6, \ldots$ are understood to be zero.