REAL FUNCTIONS

Objectives: After the completion of this section the student

- should recall the definition of the basic algebraic and transcendental functions
- should be able to determine the main properties of the functions and graph the functions

Contents:

1. Constant Function
2. Absolute Value
3. Linear Function
4. Quadratic Function
5. Polynomials
6. Rational Function
7. Irrational Functions
8. Exponential Function
9. Logarithmic Function
10. Trigonometric Functions
11. Inverse Trigonometric Functions
12. Hyperbolic Function
14. Review Questions and Exercises

REAL FUNCTIONS

A survey of elementary real-valued functions of real variable

$$f : A \subset \mathbb{R} \to \mathbb{R}$$

with their definitions and main properties is presented.
Functions can be given in *explicit* form

$$y = f(x) \qquad\qquad y = 2x+3$$

in the *implicit* form

$$f(x,y) = 0 \qquad\qquad y^3 + y - 4x = 0$$

or can be given *parametrically*

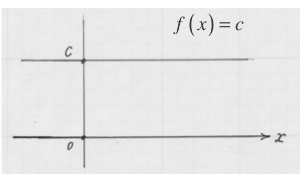
$$x = f(t)$$
$$y = g(t)$$

Functions can be also specified by their *graph* or given by a *table of values*.
Functions are called ***algebraic*** if they are polynomials, roots or rational functions, otherwise they are called ***transcendental*** functions (exponential, logarithmic, hyperbolic, trigonometric). The transcendental functions often can be defined by the infinite series.
Properties of the functions include: domain of definition, range of values, quadrant, periodicity, monotonicity, symmetry, asymptotes, characteristic particular values (zeros, poles, points of discontinuity, extremes, points of inflection).

1. CONSTANT FUNCTION:

The constant function is defined by equation

$$f(x) = c$$



It assigns the same value c for all values of variable $x \in \mathbb{R}$. The constant function is a solution of differential equation

$$\frac{d}{dx} f(x) = 0$$

Graphically, the constant function is represented by a horizontal straight line passes through the point $(0,c)$. It is defined by equation $y = c$.

2. ABSOLUTE VALUE:

The absolute value function $f(x) = |x|$ is defined as

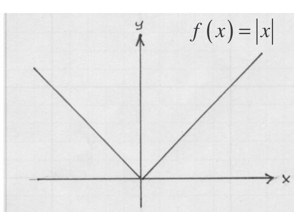
$$|x| = \begin{cases} x & if \quad x \ge 0 \\ -x & if \quad x<0 \end{cases}$$

The other definition of the absolute value function uses the root of the square

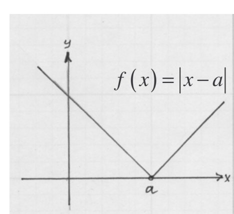
$$|x| = \sqrt{x^2}$$

It defines the distance between the points 0 and x on the real line.
Function is defined for all $x \in \mathbb{R}$. The function values are never negative, the range of values: $0 \le |x| < \infty$.

Graph of the function $f(x) = |x|$



Shifting along the x-axis:



Properties:

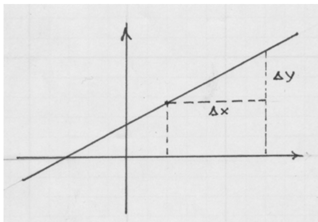
1. $|x| \ge 0$ $\qquad$ $|x| = 0$ only if $x = 0$
2. $|-x| = |x|$ $\qquad$ for all $x \in \mathbb{R}$
3. $|x-y| = |y-x|$ $\qquad$ for all $x, y \in \mathbb{R}$
4. $|x \cdot y| = |x||y|$ $\qquad$ for all $x, y \in \mathbb{R}$
5. $|x+y| \le |x| + |y|$ $\qquad$ for all $x, y \in \mathbb{R}$ $\quad$ *(triangle inequality)*

3. LINEAR FUNCTION:

A linear function is a function defined for all real numbers which describes a straight line in the plane

$$f(x) = ax + b$$

It is given by a polynomial of degree one with the following forms of equation:

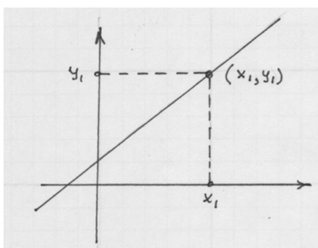


1) Slope-intercept equation:

$$y = mx + b \qquad m, b \in \mathbb{R}$$

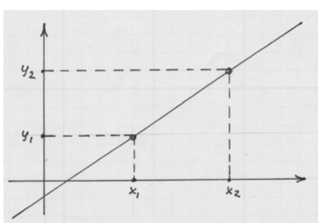
m is called the *slope* and b is called the *intercept* of the function.
The intercept b translates the function along the y-axis.

Increment of any points on the line: $m = \dfrac{\Delta y}{\Delta x}$



2) Line passes through the point (x_1, y_1) with the slope m:

$$y = m(x - x_1) + y_1$$



3) Line passes through two fixed points (x_1, y_1) and (x_2, y_2)

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

The slope m defines the inclination of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and the angle with the x-axis enclosed by the line

$$m = \tan\varphi$$

4) General linear equation:

$$Ax + By + C = 0 \qquad A, B, C \in \mathbb{R} \quad A^2 + B^2 > 0$$

4) Parametric equation of the line:

$$\begin{cases} x = t \\ y = mt + b \end{cases} \qquad -\infty < t < \infty$$

5) Differential equation of the line

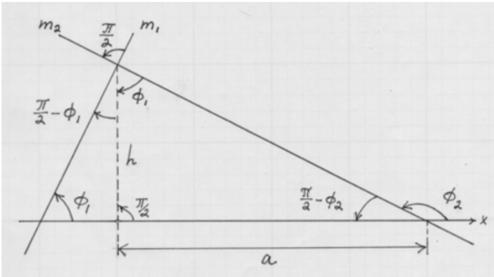
$$\frac{d}{dx} f(x) = m$$

The linear function is strictly increasing for $m > 0$
strictly decreasing for $m < 0$
a constant for $m = 0$

Perpendicular lines

If two non-vertical lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are perpendicular, then

$$m_1 = -\frac{1}{m_2}$$



Proof: $m_1 = \tan\phi_1 = \dfrac{a}{h}$

$$m_2 = \tan\phi_2 = -\tan\left(\frac{\pi}{2} - \phi_2\right) = -\frac{h}{a} = -\frac{1}{\frac{a}{h}} = -\frac{1}{m_1}$$

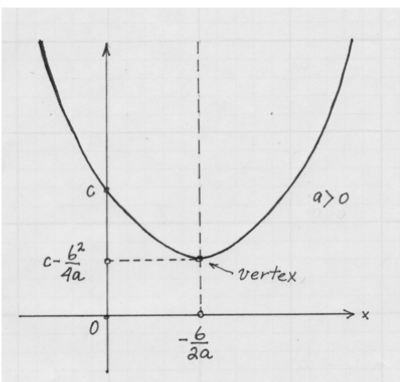
4. QUADRATIC FUNCTION: Quadratic function is a function defined for all real numbers by the equation

$$y = ax^2 + bx + c$$

This equation can be reduced to the full square form:

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$



The graph of the quadratic function is a parabola shifted to the left by $\dfrac{b}{2a}$ and up by $c - \dfrac{b^2}{4a}$.

The point $\left(-\dfrac{b}{2a}, c - \dfrac{b^2}{4a}\right)$ is called the *vertex* of the parabola.

For $a > 0$, concave up with a *global minimum* at $x = -\dfrac{b}{2a}$;

for $a < 0$, concave down with a *global maximum* at $x = -\dfrac{b}{2a}$.

The parabola is *symmetric* with respect to vertical line $x = -\dfrac{b}{2a}$.

The roots of quadratic equation $ax^2 + bx + c = 0$ determine the points of intersection of the parabola with the *x*-axis:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

If the discriminant $D \equiv b^2 - 4ac > 0$, then there are two intersections at $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$. If the discriminant $D \equiv b^2 - 4ac = 0$, then there is the intersection at $x = \dfrac{-b}{2a}$. If the discriminant $D \equiv b^2 - 4ac < 0$, then the parabola has no intersections with the *x*-axis.

Differential equation for the quadratic function:

$$\frac{d^2}{dx^2} f(x) = ax$$

Power function

The **power function** is defined all $x \in \mathbb{R}$ by the equation

$$f(x) = ax^n \qquad n = 0, 1, 2, 3, \ldots \qquad a \in \mathbb{R}$$

It is called the ***monomial function***.

5. POLYNOMIALS:

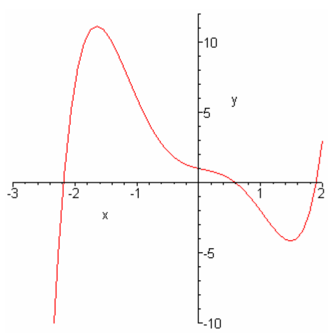
A linear combination of monoms of different powers forms the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \qquad a_1, a_2, ..., a_n \in \mathbb{R}$$

The highest power of the monoms, n, is called the ***degree*** of the polynomial. $a_1, a_2, ..., a_n \in \mathbb{R}$ are the coefficients of the polynomial, and $a_n \neq 0$ is called the *leading coefficient*. The polynomial function is defined for all real numbers $x \in \mathbb{R}$, and the range of the function depends on the special case of polynomial. For odd n, the range is $-\infty < f(x) < \infty$.

In general, the polynomial function is not symmetric. But if the polynomial contains only *even* powers, then it has mirror symmetry about the $y-$axis. And if the polynomial contains only *odd* powers then it has point symmetry about the origin.

$f(x) = x^5 - 4x^3 + x^2 - x + 1$



The graph of the polynomial function can have at most n intersections with the x-axis (real zeros). If n is odd then it has at least one real zero. It also can have up to $n-1$ extremes and up to $n-2$ points of inflection.

The linear polynomial $x - x_0$ is a factor of the polynomial function $f(x)$ if and only if x_0 is a zero of the polynomial, i.e. $f(x_0) = 0$.

According to the *Fundamental Theorem of Algebra*, the polynomial function has exactly n zeros which can be complex or real, single or repeated. Because the polynomial function has only real coefficients, then the complex zeros appear in conjugate pairs. It means that the polynomial can be represented as a product of linear factors corresponding to real zeros and irreducible quadratic factors which correspond to complex zeros. For example, the cubic polynomial $f(x) = x^3 - 2x^2 + x - 2$ has one real zero $x_1 = 2$ and two conjugate complex zeros $x_{2,3} = \pm i$. It can be factored in the following way

$$f(x) = x^3 - 2x^2 + x - 2 = (x-2)(x-i)(x+i) = (x-2)(x^2+1)$$

In general, the n^{th} the polynomial is represented as a product of linear factors

$$f(x) = a_n (x - x_1)^{s_1} (x - x_2)^{s_2} \cdots (x - x_k)^{s_k}$$

where x_i is the real or complex zero of multiplicity s_i, and k is the total number of distinct zeros.

Although the zeros of polynomial up to degree $n = 4$ can be calculated analytically, in practice only for the quadratic polynomial the analytic formula is used:

$$x_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

For polynomials of higher degree, in general, the numerical methods are used. In Maple, the polynomials can be factored with the following command:

```
> factor(x^5-x^4-x+1);
```

$$(x+1)(x^2+1)(x-1)^2$$

The roots of the polynomial function can be found by

```
> solve(x^5-4*x^3+x^2+x+1=0.,x);
```

1., -2.087922659, -0.2998974261 − 0.4402773912*I*, -0.2998974261 + 0.4402773912*I*, 1.687717512

6. RATIONAL FUNCTION: A ***rational function*** is defined as a quotient of two polynomial functions:

$$f(x) = \frac{p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0}{q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0} = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are the polynomial functions of degrees n and m, correspondingly.

If $n < m$ then the rational function is called ***proper***.
If $n > m$ then the rational function is called ***improper***.

By polynomial division (*long division*), an improper rational function can always be represented as a sum of polynomial and a proper rational function:

$$f(x) = \frac{P_n(x)}{Q_m(x)} = p_{n-m}(x) + \frac{r_l(x)}{Q_m(x)} \qquad \text{with } \; l < m$$

where the numerator polynomial $r_l(x)$ is called a ***remainder***.

By partial fraction decomposition, the remainder function $\dfrac{r_l(x)}{Q_m(x)}$ can be written as a sum of simple partial fractions (see Section 2.3.5).

Long Division

$$\frac{P_n(x)}{Q_m(x)} = \frac{p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0}{q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0}$$

$$\begin{array}{rl}
 & \dfrac{p_n}{q_m} x^{n-m} + \ldots \\
q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0 & \overline{\big)\, p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0} \\
 & p_n x^n + \dfrac{p_n q_{m-1}}{q_m} x^{n-1} + \ldots + \dfrac{p_n q_1}{q_m} x^{n-m+1} + \dfrac{p_n q_0}{q_m} x^{n-m} \\
\hline
 & \left(p_{n-1} - \dfrac{p_n q_{m-1}}{q_m} \right) x^{n-1} + \ldots \\
 & \ldots
\end{array}$$

Example: $P_4(x) = 2x^4 - x^3 + x$, $Q_2(x) = x^2 + 2x + 1$

$$\begin{array}{rll}
 & 2x^2 - 5x + 8 & \equiv p(x) \\
Q_2(x) \equiv \quad x^2 + 2x + 1 & \big)\, 2x^4 - x^3 + x & \equiv P_4(x) \\
 & 2x^4 + 4x^3 + 2x^2 & \\
\hline
 & -5x^3 - 2x^2 + x & \\
 & -5x^3 - 10x^2 - 5x & \\
\hline
 & 8x^2 + 6x & \\
 & 8x^2 + 16x + 8 & \\
\hline
 & -10x - 8 & \equiv r(x) \quad \textit{remainder}
\end{array}$$

Therefore, $$f(x) = \frac{P_4(x)}{Q_2(x)} = \frac{2x^4 - x^3 + x}{x^2 + 2x + 1} = 2x^2 - 5x + 8 + \frac{-10x - 8}{x^2 + 2x + 1}$$

The long division together with the partial fraction decomposition of the remainder function can be performed with Maple by the following command:

```
> convert((2*x^4-x^3+x)/(x^2+2*x+1),parfrac,x);
```

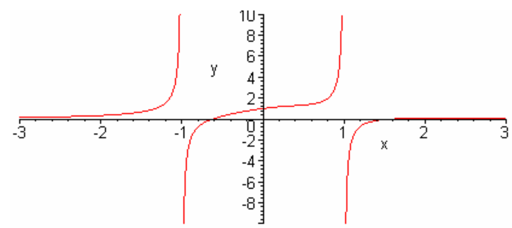
$$2\,x^2-5\,x+8-\frac{10}{x+1}+\frac{2}{(x+1)^2}$$

Graphing rational functions:

The *zeros* of the denominator polynomial $Q_m(x)$ determine the *domain* of the rational function and the poles of its graph. The particular shape of the graph, its extremes and points of inflection depend on the individual case. The *mirror symmetry* of the graph about the *y*-axis occur in the cases when both the denominator and the numerator polynomials have only even or only the odd powers. The *point symmetry* of the graph about the origin occur in the case when one of the polynomials has only even powers and the other polynomial only odd powers.

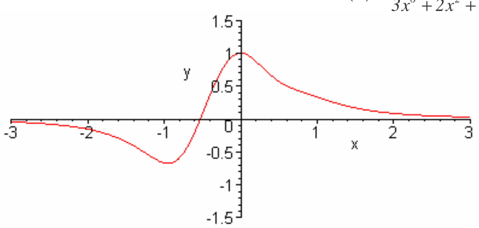
Examples:

$$f(x)=\frac{x^3-2x^2+1}{x^5-x^4-x+1}=\frac{x^3-2x^2+1}{(x+1)(x-1)^2(x^2+1)}$$



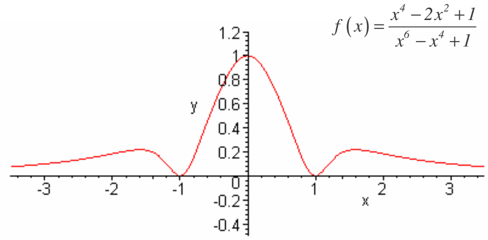
graph with poles

$$f(x)=\frac{3x^3-2x^2+1}{3x^6+2x^2+1}$$



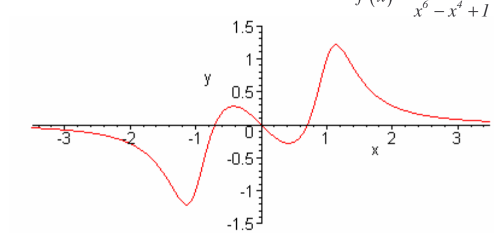
no symmetry

$$f(x)=\frac{x^4-2x^2+1}{x^6-x^4+1}$$



mirror symmetry

$$f(x)=\frac{2x^3-x}{x^6-x^4+1}$$



point symmetry

7. IRRATIONAL FUNCTIONS:

This is a wide class of functions which includes square root $y=\sqrt{x}$ and cubic root $y=\sqrt[3]{x}$ functions; root functions of integer order $y=\sqrt[n]{x}$, $n\in\mathbb{N}$; power functions with fractional exponents $y=x^{m/n}$, $m,n\in\mathbb{N}$ or the real exponent $y=x^{r}$, $r\in\mathbb{R}$; roots of polynomials and rational functions, etc. Consider some definitions of these functions:

Root function of integer order $f(x)=\sqrt[n]{x}=x^{\frac{1}{n}}$, $n\in\mathbb{N}$, $n>1$ is defined for $x\geq 0$ as an inverse of the power function:

$$y=\sqrt[n]{x} \text{ if } \quad y^{n}=x$$

and for negative exponents

$$f(x)=x^{\frac{-1}{n}}=\sqrt[n]{\frac{1}{x}}$$

For odd n, the root function is defined also for negative x values as

$$f(x)=-\sqrt[n]{-x}\,,\ x<0$$

Rules for Radicals:

Let $a,b\in\mathbb{R}$ *and* $n,m\in\mathbb{N}$*, then*

$$\sqrt[n]{1}=1 \qquad \sqrt[n]{0}=0$$

$$\sqrt[m]{a^{n}}=\left(\sqrt[m]{a}\right)^{n}=a^{\frac{m}{n}}$$

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

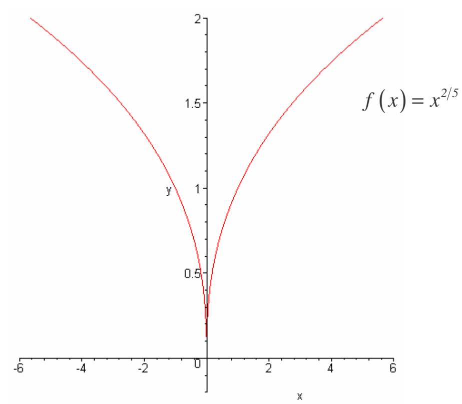
$$\sqrt[n]{\sqrt[m]{a}}=\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$$

$$\left(\sqrt[n]{a}\right)^{n}=a$$

$$\sqrt[n]{x^{n}}=\begin{cases}|x| & \text{if } n \text{ is even}\\ x & \text{if } n \text{ is odd}\end{cases},\ x\in\mathbb{R}$$

Power function of fractional order $f(x)=x^{m/n}$, $m,n\in\mathbb{Z}$ is defined as the n^{th} root of the m^{th} power

$$f(x)=x^{m/n}=\sqrt[n]{x^{m}} \qquad \text{with } n>0 \text{ and } m\neq 0$$

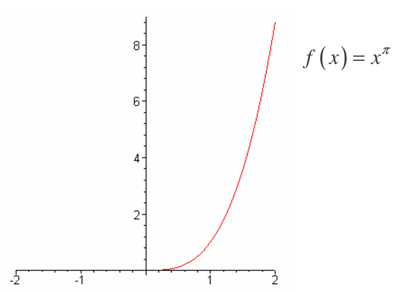


The domain of the function depends on the particular values of m and n:

$x^{\frac{m}{n}}=e^{\ln x^{\frac{m}{n}}}=e^{\ln\left(x^{m}\right)^{\frac{1}{n}}}=e^{\frac{1}{n}\ln\left(x^{m}\right)}$, then if m is even the domain is $x\neq 0$.

Power function of real order $f(x)=x^{a}$, $a\in\mathbb{R}$ is defined for $x>0$ as

$$f(x)=x^{a}=e^{a\ln x}$$



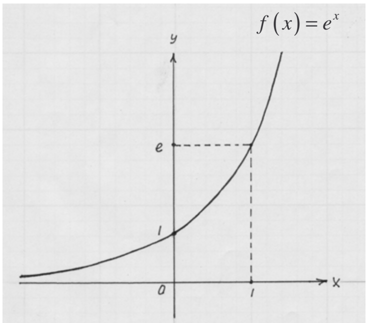
8. EXPONENTIAL FUNCTION: Let b be a positive real number not equal to 1, $b \in \mathbb{R}$, $b > 0$, $b \neq 1$. Then the function

$$f(x) = b^x$$

is an **exponential function with base b .**

If $b = e$, where e is a natural number defined as a limit $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$, then the function is called an **exponential function**:

$$f(x) = e^x$$



The relationship between functions is established by the following equation:

$$b^x = e^{lnb^x} = e^{x\,lnb} = \left(e^x\right)^{lnb}$$

where $ln\,b$ is a natural logarithm of b (see next section).

An exponential function $f(x) = e^x$ can be defined for all real x in one of the following ways:

1) Real power x of the number e:
$$e^x = inf\left\{ e^r = e^{m/n} \,\middle|\, r = m/n \in \mathbb{Q},\ r > x \right\} \text{ where } inf = infinum$$
2) Inverse of a natural logarithm function:
$$e^x = y \quad \text{if} \quad ln\,y = x, \qquad y > 0$$
3) As a limit
$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$
4) Power series (convergent for all real x)
$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$
5) As the solution of the initial value problem
$$\frac{dy}{dx} = y \qquad y(0) = 1$$

The ***general exponential function***

$$f(x) = ca^{bx+d}$$

Exponential Model The function $f(x) = ce^{kx}$, $k > 0$ models ***exponential growth*** and $f(x) = ce^{-kx}$, $k > 0$ models ***exponential decay***.

Rules for exponents: for any $a > 0, b > 0$ and for all $x, y \in \mathbb{R}$

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\left(a^x\right)^y = \left(a^y\right)^x = a^{xy}$$

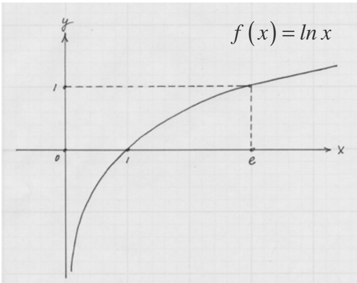
$$a^x \cdot b^x = (ab)^x$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

9. LOGARITHMIC FUNCTION: Let b be a positive real number not equal to 1: $b \in \mathbb{R},\ b > 0,\ b \neq 1$.
Then the function

$$f(x) = log_b\, x$$

is a **logarithm function with base** b (**general logarithm**).



If $b = e$, where e is a natural number defined as a limit $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$, then the function is called a **natural logarithm** (usually, logarithm without stating the base means the natural logarithm):

$$f(x) = log_e\, x = ln\, x$$

The relationship between function is established by the following equation:

$$log_b\, x = \frac{ln\, x}{ln\, b}$$

The logarithm function can be defined as inverse of the exponential function:

$y = log_b\, x$ if $b^y = x$ for all $x > 0$

$y = ln\, x$ if $e^y = x$ for all $x > 0$

Rules of logarithms: For all real $x > 0$ and $y > 0$:

$log_b\, xy = log_b\, x + log_b\, y$	$ln\, xy = ln\, x + ln\, y$	*product rule*
$log_b \frac{x}{y} = log_b\, x - log_b\, y$	$ln \frac{x}{y} = ln\, x - ln\, y$	*quotient rule*
$log_b\, x^y = y\, log_b\, x$	$ln\, x^y = y\, ln\, x$	*power rule*
$log_b\, 1 = 0$	$ln\, 1 = 0$	
$b^{log_b x} = x$	$e^{ln\, x} = x$	
$log_b\, b^x = x$	$ln\, e^x = x$	
$log_b\, b = 1$	$ln\, e = 1$	
$log_b \frac{1}{x} = -log_b\, x$	$ln \frac{1}{x} = -ln\, x$	

note! $log_b(x+y) \neq log_b\, x + log_b\, y$ $\quad ln(x+y) \neq ln\, x + ln\, y$ *(typical mistake)*

In pre-computer (pre-calculator) era, the logarithms were the main tool for performing arithmetic operations.

Conversion formulas:

$b^x = e^{x\, ln\, b}$	$log_b\, x = \frac{ln\, x}{ln\, b}$

Proof: start with $y = log_b\, x \;\Rightarrow\; x = b^y$ *definition*

$\Rightarrow\; ln\, x = ln\, b^y$ *take logarithm*

$\Rightarrow\; ln\, x = y\, ln\, b$ *power rule*

$\Rightarrow\; \dfrac{ln\, x}{ln\, b} = y$ *solve for y*

Exponential growth (decay) model $Q = Q_0 e^{\pm kt}$

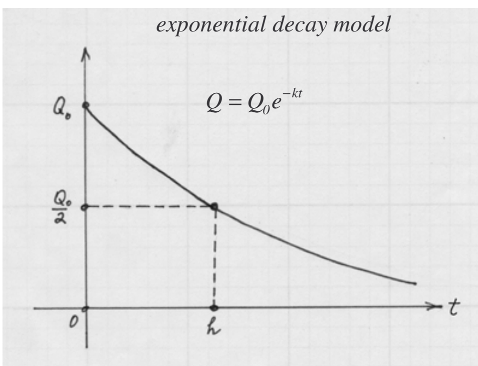
where Q_0 is the initial amount of substance at $t = 0$

1) if it is given Q_1 the amount of substance at $t = t_1$, then

$$Q_1 = Q_0 e^{kt_1} \quad\Rightarrow\quad \frac{Q_1}{Q_0} = e^{kt_1} \quad\Rightarrow\quad k = \frac{1}{t_1} ln \frac{Q_1}{Q_0}$$

Then exponential model becomes:

$$Q = Q_0 e^{kt} = Q_0 e^{\frac{t}{t_1} ln \frac{Q_1}{Q_0}} = Q_0 e^{ln\left(\frac{Q_1}{Q_0}\right)^{\frac{t}{t_1}}} = Q_0 \left(\frac{Q_1}{Q_0}\right)^{\frac{t}{t_1}}$$



exponential decay model

2) ***Half-life time*** h is defined as the time needed for substance to be reduced by a half:

$$\frac{Q_0}{2} = Q_0 e^{-kh} \qquad k > 0$$

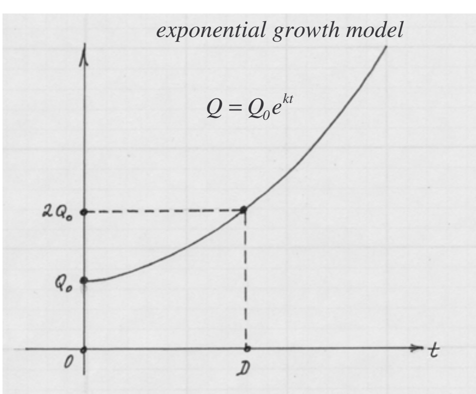
Then $\dfrac{1}{2} = e^{-kh} \;\Rightarrow\; ln\dfrac{1}{2} = ln\, e^{-kh}$

$\Rightarrow\; -ln\, 2 = -kh$

$\Rightarrow\; k = \dfrac{1}{h} ln\, 2$

Then exponential model becomes:

$$Q = Q_0 e^{-kt} = Q_0 e^{\frac{-t}{h} ln\, 2} = Q_0\, 2^{\frac{-t}{h}}$$



exponential growth model

3) ***Doubling time*** D is defined as the time needed for substance to be doubled:

$$2Q_0 = Q_0 e^{kh} \qquad k > 0$$

Then $2 = e^{-kh} \;\Rightarrow\; ln\, 2 = ln\, e^{kD}$

$\Rightarrow\; ln\, 2 = kD$

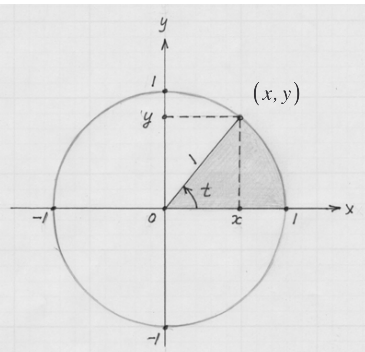
$\Rightarrow\; k = \dfrac{1}{D} ln\, 2$

Then exponential model becomes:

$$Q = Q_0 e^{kt} = Q_0 e^{\frac{t}{D} ln\, 2} = Q_0\, 2^{\frac{t}{D}}$$

10. TRIGONOMETRIC FUNCTIONS: The trigonometric functions (also called the circular functions), in calculus, are defined with the help of the unit circle (circle of radius 1). Consider a unit circle with a center placed at the origin of the Cartesian coordinates in the plane. Consider a point on the circle with the coordinates (x, y). The segment connecting the point with the origin has the unit length. From the Pythagorean Theorem follows

$$x^2 + y^2 = 1$$



This segment forms an angle with the x-axes counted positive in counter-clock direction and negative in clock direction. The angles are measured in terms of radians, where 1 radian is a measure of the angle which corresponds to the arc of length 1 on the unit circle. No units are attached to the value of angles in radians. Denote the measure of angles by the variable t.

The points of intersection of the unit circle with coordinate axes correspond to angles $t = 0$, $t = \frac{\pi}{2}$ (right angle), $t = \pi$ (stright angle), $t = \frac{3\pi}{2}$, and when we return to the first point, $t = 2\pi$ (full angle). The rotation of the point on the unit circle in the counter-clock direction defines the periodic values of angles corresponding to the same point on the unit circle

$$t = \alpha + n \cdot 2\pi \quad \text{where } n \text{ is a number of full rotations}$$

Then the set of all possible angles defined by the points on the unit circle is the set of real numbers $t \in \mathbb{R}$.

Functions $\sin t = x$, $\cos t = y$

The basic trigonometric functions are defined for all $t \in \mathbb{R}$ in the following way:

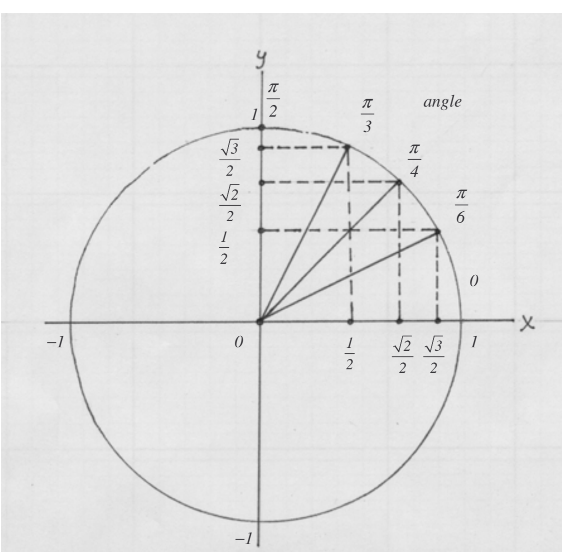
$$\boxed{\sin t = x \qquad \cos t = y}$$

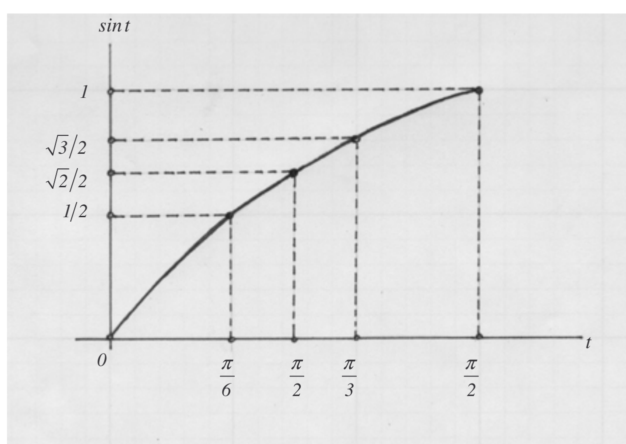
Because the same coordinates correspond to the angle after the full rotation, the introduced functions have the period $p = 2\pi$:

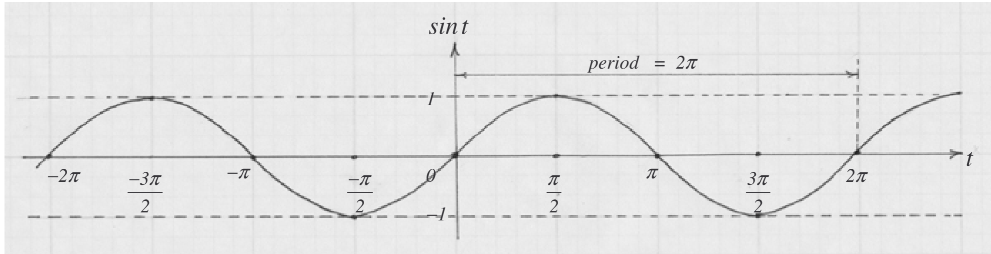
$$\sin(t + 2\pi) = \sin t \qquad \cos(t + 2\pi) = \cos t$$

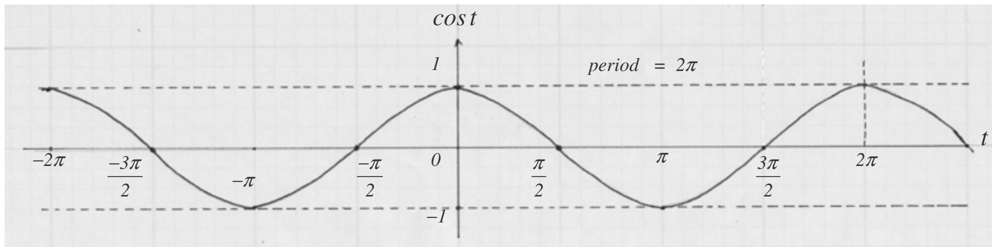
The range of both functions is between -1 and 1.

The values of functions $\sin t$ and $\cos t$ for the key angles in the first quadrant can be easily determined from the right triangles are indicated in the following graph. They also can be plotted in the graph above the interval $[0, \pi/2]$ yielding a curve which is a main element in constructing the graph of both functions by flopping and reflections:









These two graphs demonstrate that one of the functions can be obtained by shifting the other:

$$\cos t = \sin\left(t + \frac{\pi}{2}\right) \qquad \sin t = \cos\left(t - \frac{\pi}{2}\right)$$

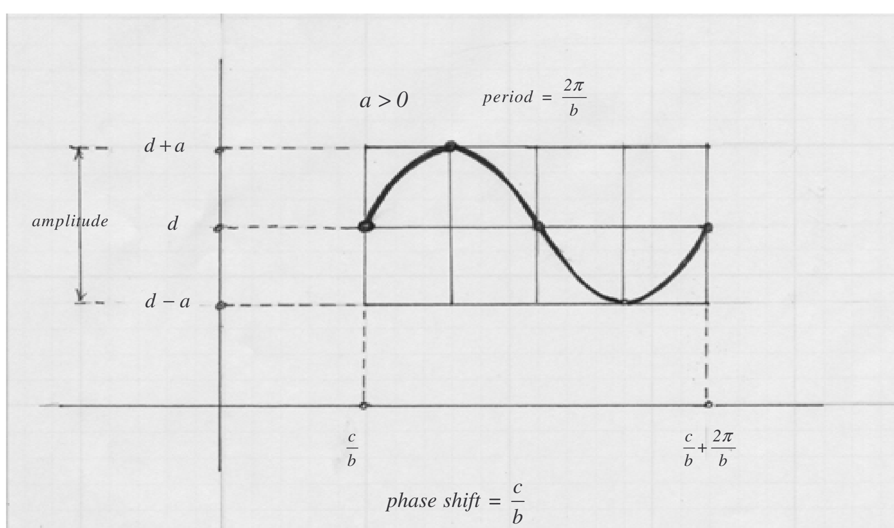
Table of the particular values:

angle t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Sinusoidal Function

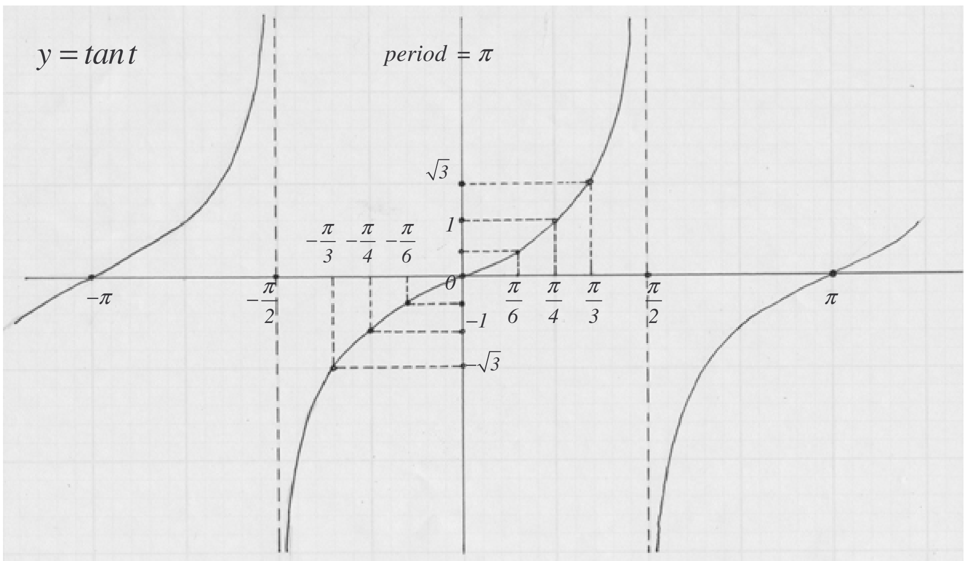
$$y = d + a\sin(bx - c) = d + a\sin\left[b\left(x - \frac{c}{b}\right)\right] \qquad period = \frac{2\pi}{b}$$

Key point method for graphing the sinusoidal function (one period):

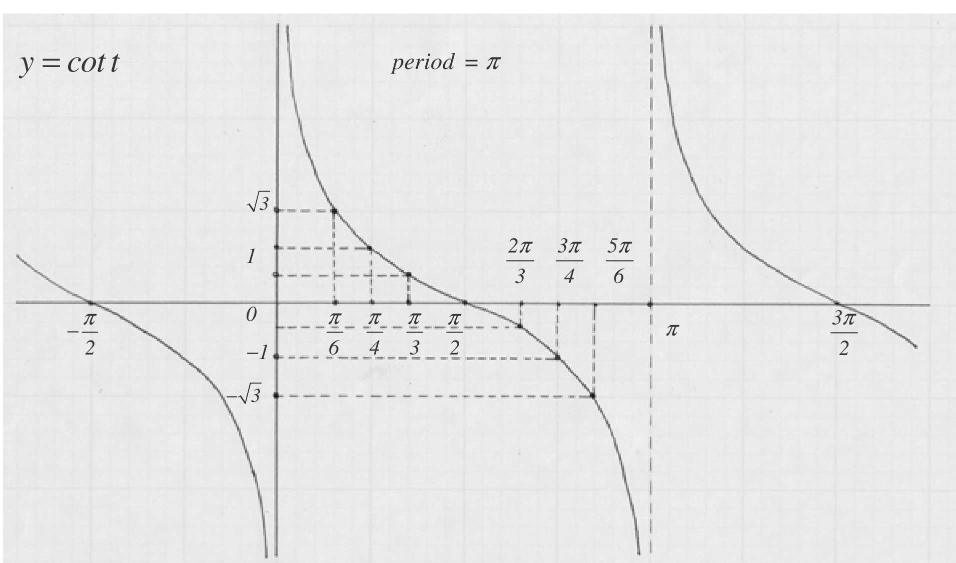


Other Trigonometric Functions: $\tan t = \dfrac{\sin t}{\cos t}$ $\cot t = \dfrac{\cos t}{\sin t}$ $\csc = \dfrac{1}{\sin t}$ $\sec t = \dfrac{1}{\cos t}$

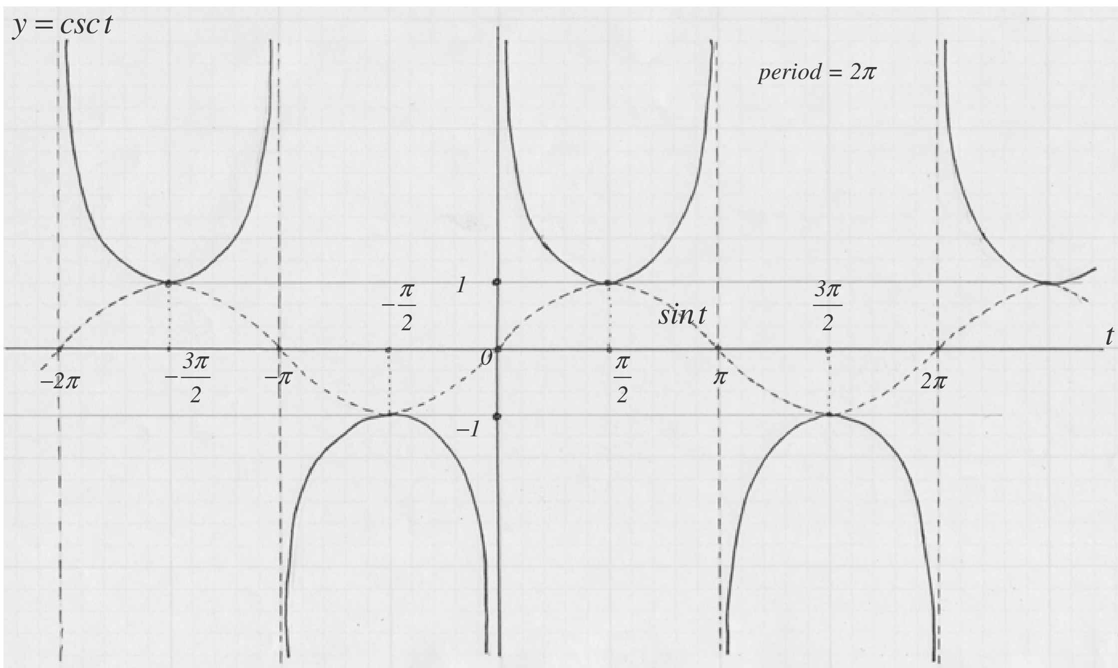
angle t	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	-90°	-60°	-45°	-30°	0°	30°	45°	60°	90°
$\tan t$	$-\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞

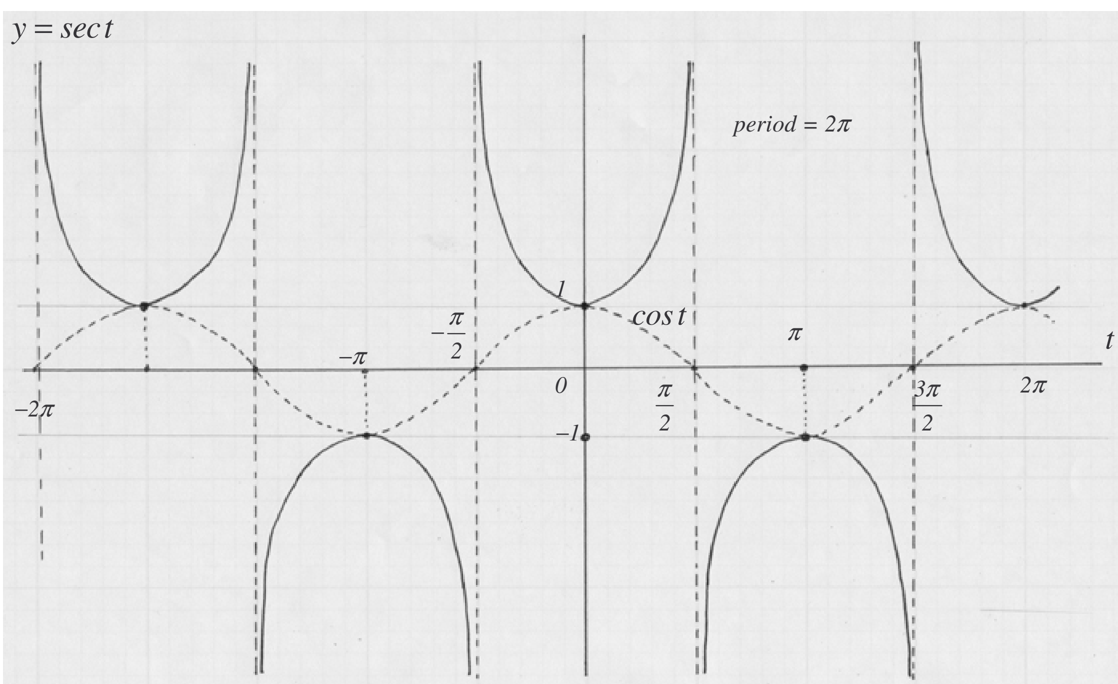


angle t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\cot t$	∞	$\sqrt{3}$	1	$-\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$-\infty$



angle t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
$\csc t$	$\pm\infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\pm\infty$	-1	$\pm\infty$
$\sec t$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\pm\infty$	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$\pm\infty$	1





Power Series Definition

The trigonometric functions also can be defined by the following infinite series convergent for all $t \in \mathbb{R}$:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

and convergent in the interval:

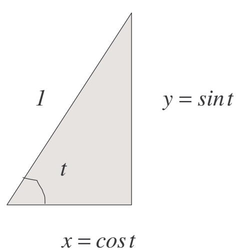
$$\tan x = \quad = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\cot x = \quad = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{x^5}{945} - \dots \qquad -\pi < x < \pi$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots \qquad -\pi < x < \pi$$

Pythagorean Identities:



The Pythagorean Theorem yields its trigonometric version

$$\boxed{\sin^2 t + \cos^2 t = 1}$$

which establishes the connection between functions:

$$\sin t = \pm\sqrt{1 - \cos^2 t} \qquad \textit{sign depends on the value of } x$$

$$\cos t = \pm\sqrt{1 - \sin^2 t}$$

By division of the Pythagorean identity by $\cos t$ and $\sin t$, consequently, one also can obtain the following formulas:

$$\boxed{\begin{aligned} 1 + \tan^2 t &= \sec^2 t \\ 1 + \cot^2 t &= \csc^2 t \end{aligned}}$$

Symmetry

$$\sin(-t) = -\sin t \qquad \textit{odd function}$$

$$\cos(-t) = \cos t \qquad \textit{even function}$$

Compliment Formulas

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t \qquad \cot\left(\frac{\pi}{2} - t\right) = \tan t$$

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \qquad \tan\left(\frac{\pi}{2} - t\right) = \cot t$$

$$\tan\left(t \pm \frac{\pi}{2}\right) = -\cot t \qquad \cot\left(t \pm \frac{\pi}{2}\right) = -\tan t$$

Angle Sum Formulas

$$\sin(t + s) = \cos t \sin s + \sin t \cos s$$

$$\sin(t - s) = \sin t \cos s - \sin s \cos t$$

$$\cos(t + s) = \cos t \cos s - \sin t \sin s$$

$$\cos(t - s) = \cos t \cos s + \sin t \sin s$$

Double Angle Formulas

$$\sin 2t = 2\sin t\cos t$$

$$\begin{aligned}\cos 2t &= \cos^2 t-\sin^2 t\\ &= 2\cos^2 t-1\\ &= 1-2\sin^2 t\end{aligned}$$

$$\tan 2t = \frac{2\tan t}{1-\tan^2 t}$$

Power Reducing Formulas

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$\tan^2 t = \frac{1-\cos 2t}{1=\cos 2t}$$

Half Angle Formulas

$$\sin\frac{t}{2} = \pm\sqrt{\frac{1-\cos t}{2}}$$

$$\cos\frac{t}{2} = \pm\sqrt{\frac{1+\cos t}{2}}$$

$$\tan\frac{t}{2} = \frac{1\text{-}\cos t}{\sin t} = \frac{\sin t}{1+\cos t}$$

Product-to-Sum

$$\sin u\sin v = \frac{1}{2}\left[\cos(u-v)-\cos(u+v)\right]$$

$$\cos u\cos v = \frac{1}{2}\left[\cos(u-v)+\cos(u+v)\right]$$

$$\sin u\cos v = \frac{1}{2}\left[\sin(u+v)+\sin(u-v)\right]$$

$$\cos u\sin v = \frac{1}{2}\left[\sin(u+v)-\sin(u-v)\right]$$

Sum-to-Product

$$\sin u+\sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u-\sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

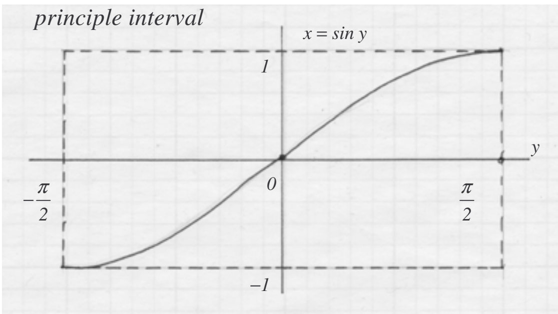
$$\cos u+\cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u-\cos v = 2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

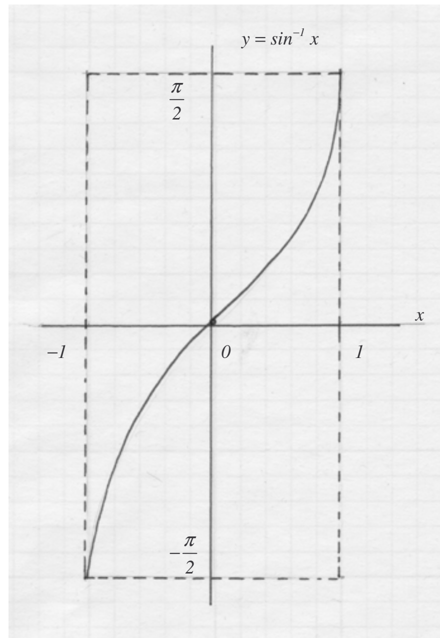
11. INVERSE TRIGONOMETRIC FUNCTIONS:

All trigonometric functions are not one-to-one (obviously, that horizontal line test fails for them). Therefore, for construction of the inverse function, we choose only the interval where a trigonometric function is one-to-one (***principle interval***) and define the inverse functions in the following way:

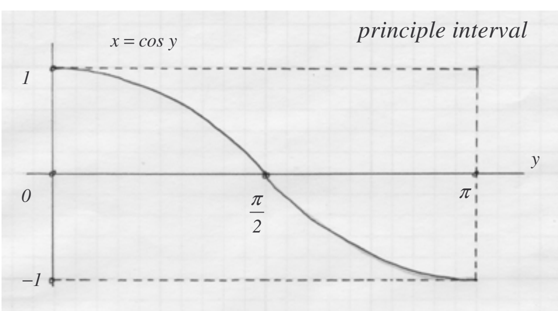
1) $y = sin^{-1} x$ if $sin\, y = x$ for all $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



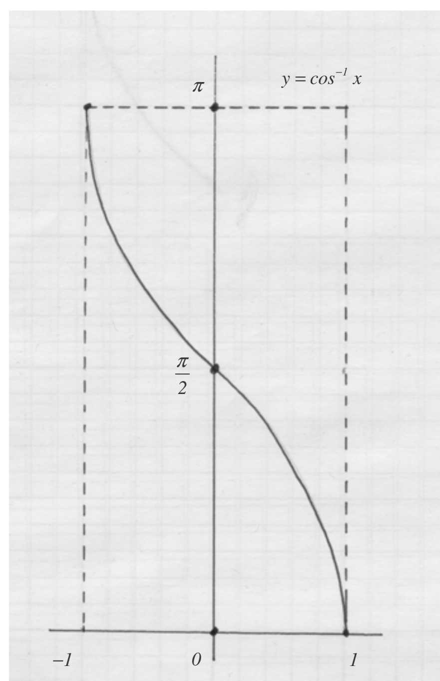
$\Rightarrow$



2) $y = cos^{-1} x$ if $cos\, y = x$ for all $0 \le y \le \pi$



$\Rightarrow$



According to definition: $sin\left(sin^{-1} x\right) = x$ $\quad sin^{-1}\left(sin\, x\right) = x$

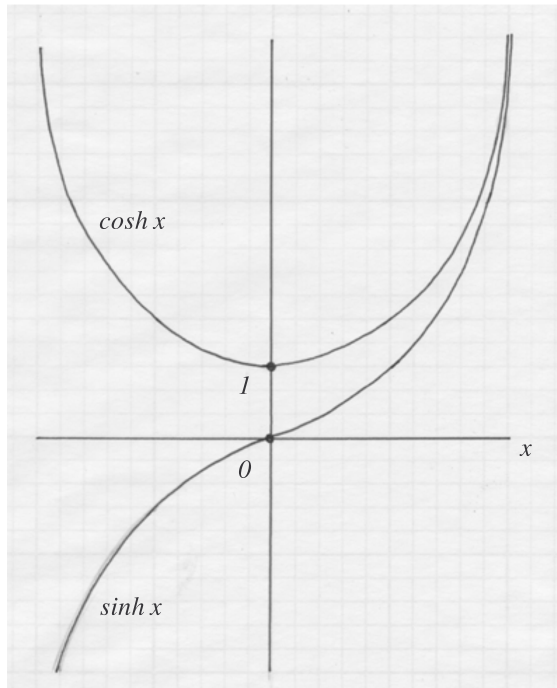
$cos\left(cos^{-1} x\right) = x$ $\quad cos^{-1}\left(cos\, x\right) = x$

12. HYPERBOLIC FUNCTIONS:

The hyperbolic functions are defined with the help of exponential functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$



Value at 0 :	$\sinh 0 = 0$	$\cosh 0 = 1$
Symmetry:	$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
Derivative:	$\sinh' x = \cosh x$	$\cosh' x = \sinh x$

Identities:

$\cosh^2 x - \sinh^2 x = 1$ *Pythagorean Identity*

$\cosh x + \sinh x = e^x$ *De Moivre's formulas*

$\cosh x - \sinh x = e^{-x}$

$\cosh nx + \sinh nx = [\cosh x + \sinh x]^n = e^{nx}$

$\cosh nx - \sinh nx = [\cosh x - \sinh x]^n = e^{-nx}$

14. REVIEW QUESTIONS:

1) By what properties the functions are characterized?
2) What are the domain and the range of the functions?
3) What functions are algebraic and functions are transcendental?

EXERCISES:

1) Sketch the graph of the ***break function*** defined with the help of the absolute value function

$$f(x)=\frac{x+|x|}{2}$$

2) Sketch the graph of the functions:

a) $f(x)=|2x-3|$ b) $f(x)=|-3x+5|$

3) Sketch the graph of the functions:

a) $f(x)=log_2 x$ b) $f(x)=log_3 5x$

c) $f(x)=log_{0.5} x$ d) $f(x)=log_{0.1} 2x$

e) $f(x)=log_4(x-5)$ f) $f(x)=log_2 x^2$

g) $f(x)=log_4(x-2)^3$ h) $f(x)=log_4(2x+3)^2$

i) $f(x)=1+e^{\frac{-x}{2}}$ j) $f(x)=2+e^{x-2}$

k) $f(x)=1+e^{\frac{-x}{2}}$ l) $f(x)=1+e^{\frac{-x}{2}}$

m) $f(x)=x^{\frac{2}{3}}$ n) $f(x)=x^{\frac{5}{2}}$

4) Prove the properties for the general power function:

a) $x^a x^b = x^{a+b}$ b) $(x^a)^b = x^{ab}$

5) Derive the trigonometric identities:

a) $sin\,3x = 3\,sin\,x - 4\,sin^3 x$ b) $cos\,3x = 4\,cos^3 x - 3\,cos\,x$

c) $sin^3 x = \frac{3\,sin\,x - sin\,3x}{4}$ d) $cos^3 x = \frac{3\,cos\,x + cos\,3x}{4}$

6) Evaluate:

a) $sin(cos^{-1} x)$ b) $cos(sin^{-1} x)$

c) $sin(cos^{-1} 3x)$ d) $cos(sin^{-1} 2x)$

7) Sketch the graph of the functions:

a) $f(x)=sin\left(2x-\frac{\pi}{4}\right)$ b) $f(x)=-cos(4x+\pi)$

c) $f(x) = 3+2\,sin(2x-\pi)$ d) $f(x) = 5+2cos\left(\frac{x}{2}-\frac{\pi}{4}\right)$

e) $f(x)=sin\,x\,cos\,x$ f) $f(x)=1-2\,sin^2 x$

g) $f(x)=sin^2 x$ h) $f(x)=cos^2 x$

i) $f(x)=x\,sin\,x$ j) $f(x)=x\,cos\,x$

k) $f(x)=tan\,2x$ l) $f(x)=cot\left(\frac{x}{2}-\pi\right)$

m) $f(x)=sec\,2x$ n) $f(x)=csc\frac{x}{2}$

o) $f(x)=|\sin x|$ p) $f(x)=\left|\cos\frac{x}{2}\right|$

r) $f(x)=\sin|2x|$ s) $f(x)=\ln\left|\frac{x}{2}\right|$

i) $f(x)=x+\sin x$ j) $f(x)=2x+\ln x$

8) a) Find an exponential function $y=ae^{bx}$ the graph of which passes two fixed points $(1,2)$ and $(2,10)$. Using properties of exponential and logarithmic functions simplify the expression and sketch the graph.

b) Find an exponential function $y=ae^{bx}$ the graph of which passes two fixed points $(1,2)$ and $(2,1)$. Using properties of exponential and logarithmic functions simplify the expression and sketch the graph.

9) Express the rational function as a sum of the polynomial and a proper rational function and sketch the graph:

a) $f(x)=\frac{x^3-2x^2+x-4}{x^2+2x-1}$ b) $f(x)=\frac{x^4+3x^2-2x+1}{x^3+2x^2-1}$

10) a) At any moment of time, the rate of production of a certain biological substance is described by the exponential growth model $Q(t)=Q_0e^{kt}$. If after 1 hour there is $2\ lb$ of the substance and after 2 hours the amount is $8\ lb$, how much of the substance will be there after 3 hours of production?

b) At any moment of time, the rate of production of a certain biological substance is described by the exponential growth model $Q(t)=Q_0e^{kt}$. If after 1 hour there is $2\ lb$ of the substance and after 2 hours the amount is $10\ lb$, how much of the substance was there initially?

c) At any moment of time, the rate of fission of a certain substance is described by the exponential decay model $Q(t)=Q_0e^{-kt}$.
The half-life time is known to be 2 hours. If after 1 hour there is $10\ lb$ of the substance, how much of the substance was there initially?

11) Sketch the graph of the functions:

a) $f(x)=\cosh(2x)-\sinh(2x)$ b) $f(x)=\sinh(2x)+\cosh(2x)$

c) $f(x)=\tanh x\equiv\frac{\sinh x}{\cosh x}$ d) $f(x)=\coth x\equiv\frac{\cosh x}{\sinh x}$

f) $f(x)=\operatorname{sech} x\equiv\frac{1}{\cosh x}$ g) $f(x)=\operatorname{csch} x\equiv\frac{1}{\sinh x}$

12) Derive the identities:

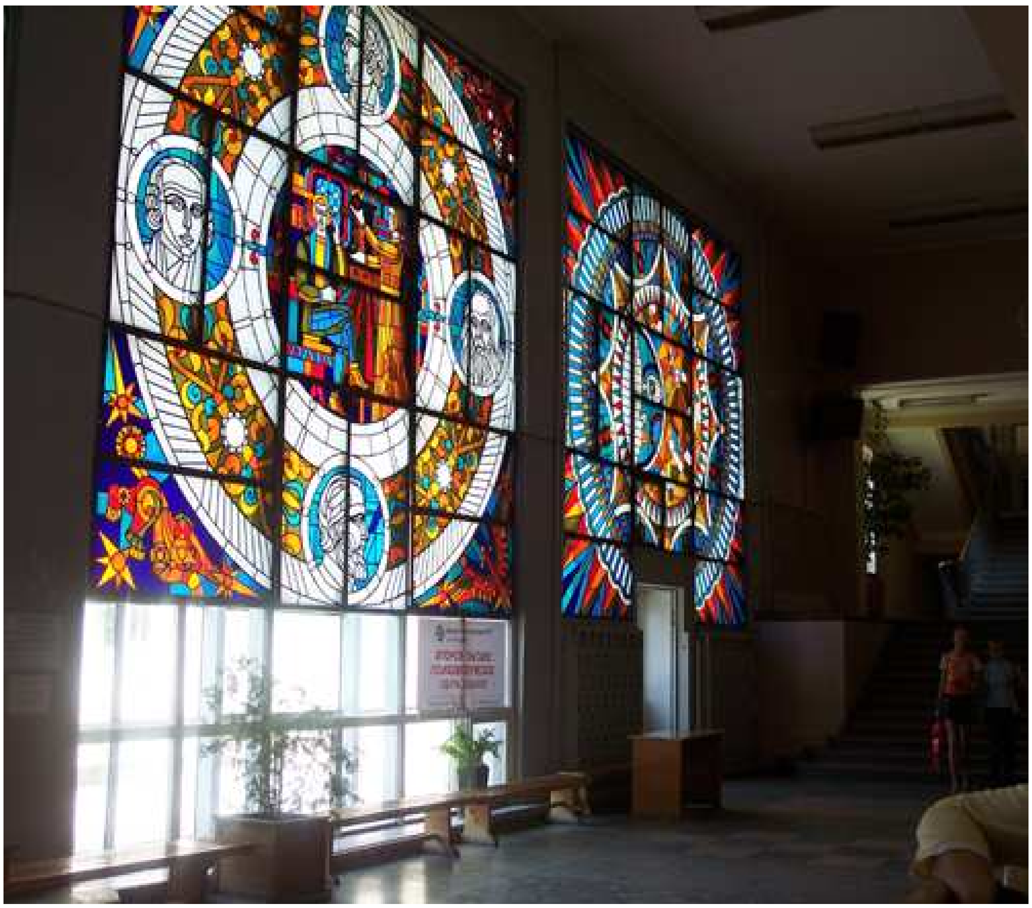
a) $\cosh^2 x-\sinh^2 x=1$ b) $1-\tanh^2 x=\operatorname{sech}^2 x$

13) Find the inverse of the functions and sketch the graph of both of them:

a) $f(x)=\frac{x+1}{2x+3}$ b) $f(x)=\frac{1}{x+2}$

c) $f(x)=2x-1$ d) $f(x)=x^3+1$

f) $f(x)=\sin\frac{x}{2}$ g) $f(x)=\sqrt{x-1}$



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