

The Laplace Equation: 05 – RDDN (Robin- Dirichlet-Dirichlet-Neumann)

Dirichlet
 $u = f_2(x) = x(L - x)$

Dirichlet
 $u = f_3(y) = y(M - y)$

Neumann
 $\left. \frac{\partial u}{\partial x} \right|_{x=L} = f_4(y) = 4$

Robin
 $\left[-k_1 \frac{\partial u}{\partial y} + h_1 u \right]_{y=0} = f_1(x) = 1$

$\nabla^2 u = 0$

↓

Supplemental problems:

0

$f_2(x)$

0

0

0

$f_3(y)$

0

0

0

$f_4(y)$

$\nabla^2 u_1 = 0$

$\nabla^2 u_2 = 0$

$\nabla^2 u_3 = 0$

$\nabla^2 u_4 = 0$

$f_1(x)$

0

0

0

0

Solution of supplemental problems:

$u_1(x, y) = \sum_{n=0}^{\infty} a_n \sin(\lambda_n x) \sinh(\lambda_n (y - M))$

Where $\lambda_n = (n + \frac{1}{2}) \frac{\pi}{L}$

$a_n = \frac{-\frac{2}{L} \int_0^L f_1(x) \sin(\lambda_n x) dx}{k_1 \lambda_n \cosh(\lambda_n M) + h_1 \sinh(\lambda_n M)}$

$u_2(x, y) = \sum_{n=0}^{\infty} b_n \sin(\lambda_n x) \left[\cosh(\lambda_n y) + \frac{h_1}{k_1 \lambda_n} \sinh(\lambda_n y) \right]$

Where $\lambda_n = (n + \frac{1}{2}) \frac{\pi}{L}$

$b_n = \frac{\frac{2}{L} \int_0^L f_2(x) \sin(\lambda_n x) dx}{\cosh(\lambda_n M) + \frac{h_1}{k_1 \lambda_n} \sinh(\lambda_n M)}$

$u_3(x, y) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n (y - M)) \cosh(\lambda_n (x - L))$

Where λ_n are the positive roots of $\lambda \cos(\lambda M) + H_1 \sin(\lambda M) = 0$

$c_n = \frac{\int_0^M f_3(y) \sin(\lambda_n (y - M)) dy}{\left(\frac{M}{2} - \frac{\sin(2\lambda_n M)}{4\lambda_n} \right) \cosh(\lambda_n L)}$

$u_4(x, y) = \sum_{n=1}^{\infty} d_n \sinh \frac{n\pi}{M} x \sin \frac{n\pi}{M} y$

$d_n = \frac{\frac{2}{M} \int_0^M f_4(y) \sin \frac{n\pi}{M} y dy}{\sinh \frac{n\pi}{M} L}$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

POISSON'S EQUATION: 01 – DDDD

A rectangular domain with vertices labeled 0 (bottom-left), L (bottom-right), M (top-left), and an unlabeled top-right corner. The boundary conditions are: $u = f_2(x)$ on the top edge, $u = f_4(y)$ on the right edge, $u = f_1(x)$ on the bottom edge, and $u = f_3(y)$ on the left edge. Inside the rectangle, the equation $\nabla^2 u = F(x, y)$ is written.

Supplemental problems:

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Two rectangular domains are shown side-by-side. The left rectangle has boundary conditions $f_2(x)$ (top), $f_4(y)$ (right), $f_1(x)$ (bottom), and $f_3(y)$ (left). Inside, the equation is $\nabla^2 u_5 = 0$. The right rectangle has boundary conditions 0 (top), 0 (right), 0 (bottom), and 0 (left). Inside, the equation is $\nabla^2 u_6 = F$.

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u_5(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u_6(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{M}y\right)$$

$$A_{nm} = \frac{-4}{\pi^2 LM \left(\frac{n^2}{L^2} + \frac{m^2}{M^2}\right)} \int_0^L \int_0^M F(x, y) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{M}y\right) dx dy$$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x, y) = u_5(x, y) + u_6(x, y)$$