

The Laplace Equation: 06 – DDNR (Dirichlet- Dirichlet-Neumann-Robin)

$$\begin{array}{ccc}
 & \text{Dirichlet} & \\
 & u = f_2(x) & \\
 M & \begin{array}{c} \text{Neumann} \\ \frac{\partial}{\partial x} u(x, y) = f_3(x) \end{array} & \begin{array}{c} \text{Robin} \\ \left(\frac{\partial}{\partial x} u(x, y) \right) + H_4 u = f_4(y) \end{array} \\
 & \nabla^2 u = 0 & \\
 0 & & L \\
 & \text{Dirichlet} & \\
 & u = f_1(x) &
 \end{array}$$

Supplemental problems:

$$\begin{array}{cccc}
 0 & f_2(x) & 0 & 0 \\
 \begin{array}{c} 0 \\ \nabla^2 u_1 = 0 \\ f_1(x) \end{array} & \begin{array}{c} 0 \\ \nabla^2 u_2 = 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ \nabla^2 u_3 = 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ \nabla^2 u_4 = 0 \\ 0 \end{array} \\
 f_1(x) & 0 & f_3(y) & f_4(y)
 \end{array}$$

Solution of supplemental problems:

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) \sinh \lambda_n (y - M)$$

$$a_n = \frac{-\int_0^L f_1(x) \cos \lambda_n x dx}{\left(\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n}\right) \sinh \lambda_n M}$$

$$\lambda_n \rightarrow \text{positive_roots_of} : \lambda \sin \lambda L - H_4 \cos \lambda L = 0$$

$$u_2(x, y) = \sum_{n=1}^{\infty} b_n \cos \lambda_n x \sinh \lambda_n y$$

$$b_n = \frac{\int_0^L f_2(x) \cos \lambda_n x dx}{\left(\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n}\right) \sinh \lambda_n M}$$

$$\lambda_n \rightarrow \text{positive_roots_of} : \lambda \sin \lambda L - H_4 \cos \lambda L = 0$$

$$u_3(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{M} y \left[\cosh \frac{n\pi}{M} (x - L) - \frac{H_4}{n\pi} \sinh \frac{n\pi}{M} (x - L) \right]$$

$$c_n = \frac{-\frac{2}{M} \int_0^M f_3(y) \sin \frac{n\pi}{M} y dy}{\frac{n\pi}{M} \sinh \frac{n\pi}{M} L + H_4 \cosh \frac{n\pi}{M} L}$$

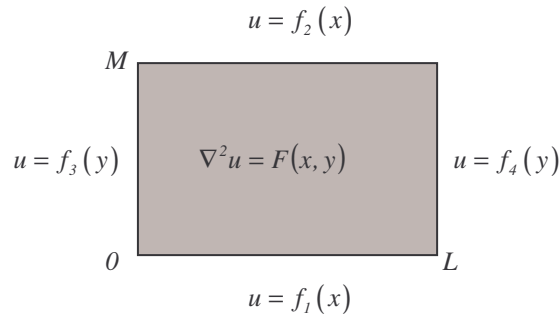
$$u_4(x, y) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{M} y \cosh \frac{n\pi}{M} x$$

$$d_n = \frac{\frac{2}{M} \int_0^M f_4(y) \sin \frac{n\pi}{M} y dy}{\frac{n\pi}{M} \sinh \frac{n\pi}{M} L + H_4 \cosh \frac{n\pi}{M} L}$$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

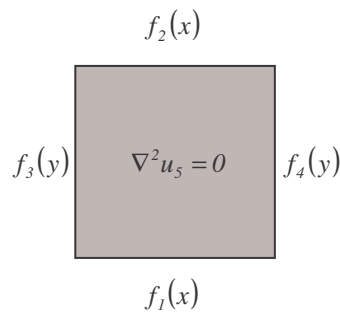
POISSON'S EQUATION: 01 – DDDD



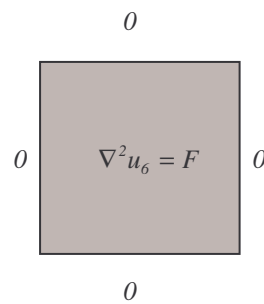
$$\nabla^2 u = F(x, y)$$

Supplemental problems:

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$$\nabla^2 u_5 = 0$$



$$\nabla^2 u_6 = F$$

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u_5(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi}{M} y \cos \lambda_m x \quad \lambda_m \rightarrow \text{positive_roots_of} : \lambda \sin \lambda L - H_4 \cos \lambda L = 0$$

$$A_{mn} = \frac{-\frac{2}{M}}{[\lambda_m^2 + (\frac{n\pi}{M})^2](\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n})} \int_0^M \int_0^L F(x, y) \sin \frac{n\pi}{M} y \cos \lambda_m x dx dy$$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x, y) = u_5(x, y) + u_6(x, y)$$