

The Laplace Equation: 07 – DNRN (Dirichlet- Neumann-Robin- Neumann)

$$\begin{array}{ccc}
 & \text{Neumann} & \\
 & \frac{\partial}{\partial x} u(x, y) = f_2(x) & \\
 \text{Robin} & \begin{array}{c} M \\ \square \\ 0 \end{array} & \text{Neumann} \\
 \left(\frac{\partial}{\partial x} u(x, y) \right) + H_3 u = f_3(y) & \nabla^2 u = 0 & \frac{\partial}{\partial x} u(x, y) = f_4(x) \\
 & L & \\
 & u = f_1(x) & \\
 & \text{Dirichlet} &
 \end{array}$$

Supplemental problems:

$$\begin{array}{cccc}
 \begin{array}{c} 0 \\ \square \\ f_1(x) \end{array} & \begin{array}{c} f_2(x) \\ \square \\ 0 \end{array} & \begin{array}{c} 0 \\ \square \\ 0 \end{array} & \begin{array}{c} 0 \\ \square \\ 0 \end{array} \\
 \nabla^2 u_1 = 0 & \nabla^2 u_2 = 0 & \nabla^2 u_3 = 0 & \nabla^2 u_4 = 0 \\
 f_1(x) & 0 & f_3(y) & f_4(y)
 \end{array}$$

Solution of supplemental problems:

$$\begin{aligned}
 u_1(x, y) &= \sum_{n=1}^{\infty} a_n \cos \lambda_n (x - L) \cosh \lambda_n (y - M) & a_n &= \frac{\int_0^L f_1(x) \cos \lambda_n (x - L) dx}{\left(\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n} \right) \cosh \lambda_n M} \\
 \lambda_n &\rightarrow \text{positive_roots_of: } \lambda \sin \lambda L - H_3 \cos \lambda L = 0 \\
 u_2(x, y) &= \sum_{n=1}^{\infty} b_n \cos \lambda_n (x - L) \sinh \lambda_n y & b_n &= \frac{\int_0^L f_2(x) \cos \lambda_n (x - L) dx}{\lambda_n \left(\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n} \right) \cosh \lambda_n M} \\
 \lambda_n &\rightarrow \text{positive_roots_of: } \lambda \sin \lambda L - H_3 \cos \lambda L = 0 \\
 u_3(x, y) &= \sum_{n=1}^{\infty} c_n \cosh \lambda_n (x - L) \sin \lambda_n y & c_n &= \frac{-\frac{2}{M} \int_0^M f_3(y) \sin \lambda_n y dy}{\lambda_n \sinh \lambda_n L - H_3 \cosh \lambda_n L} \\
 \lambda_n &= (n + 1/2) \frac{\pi}{M}
 \end{aligned}$$

$$u_4(x, y) = \sum_{n=1}^{\infty} d_n \sin \lambda_n y \left[\cosh \lambda_n x - \frac{H_3}{\lambda_n} \sinh \lambda_n x \right]$$

$$\lambda_n = (n + 1/2) \frac{\pi}{M}$$

$$d_n = \frac{\frac{2}{M} \int_0^M f_4(y) \sin \lambda_n y dy}{\lambda_n \sinh \lambda_n L - H_3 \cosh \lambda_n L}$$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

POISSON'S EQUATION: 01 – DDDD

$$\begin{array}{c} u = f_2(x) \\ M \\ u = f_3(y) \quad \nabla^2 u = F(x, y) \quad u = f_4(y) \\ 0 \quad L \\ u = f_1(x) \end{array}$$

Supplemental problems:

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$$\begin{array}{cc} \begin{array}{c} f_2(x) \\ \square \\ f_3(y) \quad \nabla^2 u_5 = 0 \quad f_4(y) \\ f_1(x) \end{array} & \begin{array}{c} 0 \\ \square \\ 0 \quad \nabla^2 u_6 = F \quad 0 \\ 0 \end{array} \end{array}$$

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u_5(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \lambda_m y \cos \lambda_n (x - L) \quad \lambda_m = (n + 1/2) \frac{\pi}{M}$$

$$A_{mn} = \frac{-\frac{2}{M}}{[\lambda_m^2 + (\lambda_n)^2] \left(\frac{L}{2} + \frac{\sin 2\lambda_n L}{4\lambda_n} \right)} \int_0^M \int_0^L F(x, y) \sin \lambda_m y \cos \lambda_n (x - L) dx dy$$

$$\lambda_n \rightarrow \text{positive_roots_of} : \lambda \sin \lambda L - H_3 \cos \lambda L = 0$$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x, y) = u_5(x, y) + u_6(x, y)$$