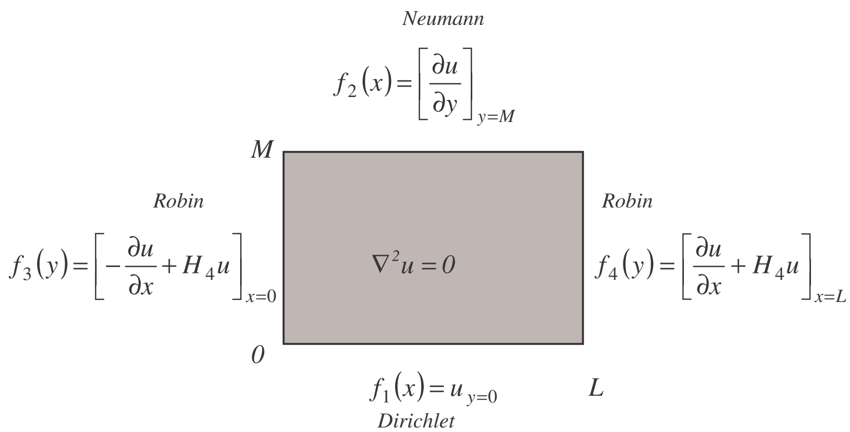
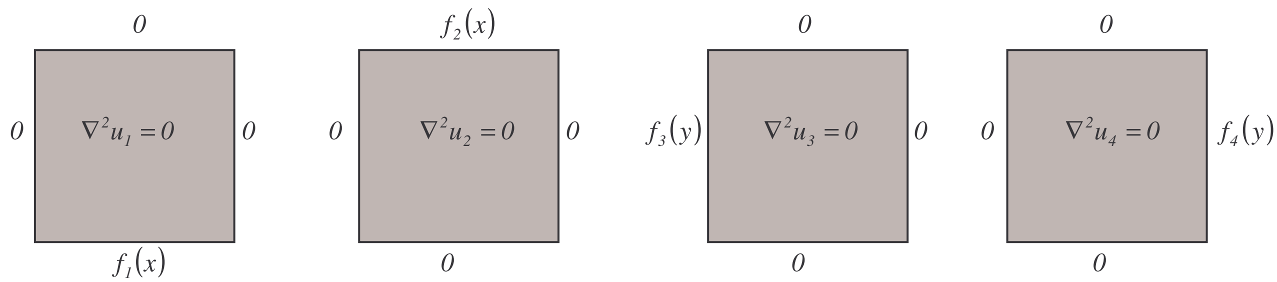
The Laplace Equation: 10 – DNRR (Dirichlet- Neumann- Robin- Robin)



Supplemental problems:



Solution of supplemental problems:

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \cosh(\lambda_n (y-M))(\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x))$$

$$a_n = \frac{\int_0^L f_1(x)(\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x))}{\left(\frac{\lambda_n^2 + H_3^2}{2}\left(L + \frac{H_4}{\lambda_n^2 + H_4^2}\right) + \frac{H_3}{2}\right)\cosh(\lambda_n M)}$$

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \cosh(\lambda_n y)(\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x))$$

$$b_n = \frac{\int_0^L f_2(x)(\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x))}{\left(\frac{\lambda_n^2 + H_3^2}{2}\left(L + \frac{H_4}{\lambda_n^2 + H_4^2}\right) + \frac{H_3}{2}\right)\lambda_n \sinh(\lambda_n M)}$$

λ_n are positive roots of

$$\left(H_3 H_4 - \lambda^2\right)\sin(\lambda L) + (H_3 + H_4)\lambda \cos(\lambda L) = 0$$

…continued solution of supplemental problems:

$$u_3(x, y) = \sum_{m=1}^{\infty} c_m (\sin(\mu_m y))(\cosh(\mu_m (x-L)) - H_3 \sinh(\mu_m (x-L)))$$

$$c_m = \frac{\int_0^L f_3(y)\sin(\mu_m y)}{\left(\frac{M}{2}\right)\left((\mu_m + H_3^2)\sinh(\mu_m L) + (H_3\mu_m + H_3)\cosh(\mu_m L)\right)}$$

$$u_4(x, y) = \sum_{m=1}^{\infty} d_m (\sin(\mu_m y))(\cosh(\mu_m x) + H_4 \sinh(\mu_m x))$$

$$d_m = \frac{\int_0^L f_4(y)\sin(\mu_m y)}{\left(\frac{M}{2}\right)\left((\mu_m + H_4^2)\sinh(\mu_m L) + (H_4\mu_m + H_4)\cosh(\mu_m L)\right)}$$

$$\mu_m = \left(m + \frac{1}{2}\right)\frac{\pi}{M}$$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

POISSON'S EQUATION 10) DNRR

Neumann

$$f_2(x)=\left[\frac{\partial u}{\partial y}\right]_{y=M}$$

Robin

$$f_3(y)=\left[-\frac{\partial u}{\partial x}+H_4 u\right]_{x=0}$$

$$\nabla^2 u = 0 \quad \text{on } 0 \le x \le L,\ 0 \le y \le M$$

Robin

$$f_4(y)=\left[\frac{\partial u}{\partial x}+H_4 u\right]_{x=L}$$

$$f_1(x)=u_{y=0}$$

Dirichlet

↓

Supplemental problems:

$\nabla^2 u_5 = 0$ with boundary values $f_2(x)$ (top), $f_3(y)$ (left), $f_4(y)$ (right), $f_1(x)$ (bottom)

$\nabla^2 u_6 = F$ with boundary values 0 (top), 0 (left), 0 (right), 0 (bottom)

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u_6(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\left(\lambda_n\cos(\lambda_n x)+H_3\sin(\lambda_n x)\right)\left(\sin(\mu_m y)\right)$$

$$A_{nm}=\frac{\int_0^M\int_0^L F(x,y)\left(\lambda_n\cos(\lambda_n x)+H_3\sin(\lambda_n x)\right)\left(\sin(\mu_m y)\right)dxdy}{\left(\frac{\lambda_n^2+H_3^2}{2}\left(L+\frac{H_4}{\lambda_n^2+H_3^2}\right)+\frac{H_3}{2}\right)\left(\frac{L}{2}\right)\left(\lambda_n^2+\mu_m^2\right)}$$

λ_n are positive roots of

$$\left(H_3H_4-\lambda^2\right)\sin(\lambda L)+\left(H_3+H_4\right)\lambda\cos(\lambda L)=0$$

$$\mu_m=\left(m+\frac{1}{2}\right)\frac{\pi}{M}$$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x,y)=u_5(x,y)+u_6(x,y)$$