

The Laplace Equation: 01 – RRRR Robin- Robin - Robin - Robin)

$$\begin{array}{ccc}
 \text{Robin} & & \text{Robin} \\
 \left[-\frac{\partial u}{\partial y} + H_2 u \right]_{y=M} & = f_2(x) & \\
 \\
 & \begin{array}{c} M \\ \hline \square \\ \hline 0 \end{array} & \\
 \begin{array}{c} \text{Robin} \\ \left[-\frac{\partial u}{\partial x} + H_3 u \right]_{x=0} = f_3(y) \end{array} & \nabla^2 u = 0 & \begin{array}{c} \text{Robin} \\ \left[-\frac{\partial u}{\partial x} + H_4 u \right]_{x=L} = f_4(y) \end{array} \\
 \\
 & \begin{array}{c} \text{Robin} \\ \left[-\frac{\partial u}{\partial y} + H_1 u \right]_{y=0} = f_1(x) \end{array} & L
 \end{array}$$

Supplemental problems:

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$$\begin{array}{cccc}
 \begin{array}{c} 0 \\ \square \\ f_1(x) \end{array} & \begin{array}{c} f_2(x) \\ \square \\ 0 \end{array} & \begin{array}{c} 0 \\ \square \\ f_3(y) \end{array} & \begin{array}{c} 0 \\ \square \\ f_4(y) \end{array} \\
 \nabla^2 u_1 = 0 & \nabla^2 u_2 = 0 & \nabla^2 u_3 = 0 & \nabla^2 u_4 = 0
 \end{array}$$

Solution of supplemental problems:

$$\begin{aligned}
 u_1(x, y) &= \sum_{n=1}^{\infty} a_n \left(\cosh(\lambda_n (y - M)) - \frac{H_4}{\lambda_n} \sinh(\lambda_n (y - M)) \right) (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) \\
 a_n &= \frac{\int_0^L f_1(x) (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) dx}{\left[\frac{1}{2} (\lambda_n^2 + H_3^2) \left(L + \frac{H_4}{(\lambda_n^2 + H_4^2)} + \frac{H_3}{2} \right) \right] \left[\left(\lambda_n + \frac{H_3 H_4}{\lambda_n} \right) \sinh \lambda_n M + (H_3 + H_4) \cosh \lambda_n M \right]} \\
 u_2(x, y) &= \sum_{n=1}^{\infty} b_n \left(\cosh(\lambda_n (y)) + \frac{H_3}{\lambda_n} \sinh(\lambda_n (y)) \right) (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) \\
 b_n &= \frac{\int_0^L f_2(x) (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) dx}{\left[\frac{1}{2} (\lambda_n^2 + H_3^2) \left(L + \frac{H_4}{(\lambda_n^2 + H_4^2)} + \frac{H_3}{2} \right) \right] \left[\left(\lambda_n + \frac{H_3 H_4}{\lambda_n} \right) \sinh \lambda_n M + (H_3 + H_4) \cosh \lambda_n M \right]}
 \end{aligned}$$

Where λ_n are positive roots of: $(H_3 H_4 - \lambda_n^2) \sin \lambda_n L + (H_3 + H_4) \lambda_n \cos \lambda_n L = 0$

$$u_3(x, y) = \sum_{n=1}^{\infty} c_n \left(\cosh(\lambda_n(x-L)) - \frac{H_2}{\lambda_n} \sinh(\lambda_n(x-L)) \right) (\lambda_n \cos(\lambda_n y) + H_1 \sin(\lambda_n y))$$

$$c_n = \frac{\int_0^M f_3(y) (\lambda_n \cos(\lambda_n y) + H_1 \sin(\lambda_n y)) dy}{\left[\frac{1}{2} (\lambda_n^2 + H_1^2) \left(L + \frac{H_2}{(\lambda_n^2 + H_2^2)} + \frac{H_1}{2} \right) \left[\left(\lambda_n + \frac{H_1 H_2}{\lambda_n} \right) \sinh \lambda_n M + (H_1 + H_2) \cosh \lambda_n M \right] \right]}$$

$$u_4(x, y) = \sum_{n=1}^{\infty} d_n \cosh(\lambda_n(x)) + \frac{H_1}{\lambda_n} \sinh(\lambda_n(x)) (\lambda_n \cos(\lambda_n y) + H_1 \sin(\lambda_n y))$$

$$d_n = \frac{\int_0^M f_4(y) (\lambda_n \cos(\lambda_n y) + H_1 \sin(\lambda_n y)) dy}{\left[\frac{1}{2} (\lambda_n^2 + H_1^2) \left(L + \frac{H_2}{(\lambda_n^2 + H_2^2)} + \frac{H_1}{2} \right) \left[\left(\lambda_n + \frac{H_1 H_2}{\lambda_n} \right) \sinh \lambda_n M + (H_1 + H_2) \cosh \lambda_n M \right] \right]}$$

Where λ_n are positive roots of: $(H_1 H_2 - \lambda_n^2) \sin \lambda_n M + (H_1 + H_2) \lambda_n \cos \lambda_n M = 0$

Solution of BVP problem:

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

POISSON'S EQUATION: 01 – RRRR

$$u = f_2(x)$$

$$u = f_3(y)$$

$$\nabla^2 u = F(x, y)$$

$$u = f_4(y)$$

$$u = f_1(x)$$

Supplemental problems:

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$$f_2(x)$$

$$f_3(y)$$

$$\nabla^2 u_5 = 0$$

$$f_4(y)$$

$$f_1(x)$$

$$0$$

$$0$$

$$\nabla^2 u_6 = F$$

$$0$$

$$0$$

Solution of supplemental problems:

Solution of Laplace's homogeneous equation with non-homogeneous b.c.'s:

$$u_5(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

Solution of Poisson's equation with homogeneous boundary conditions

$$u_6(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{n,m} (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) (\lambda_m \cos(\lambda_m y) + H_1 \sin(\lambda_m y))$$

$$A_{n,m} = \frac{\int_0^M \int_0^L F(x, y) (\lambda_n \cos(\lambda_n x) + H_3 \sin(\lambda_n x)) (\lambda_m \cos(\lambda_m y) + H_1 \sin(\lambda_m y)) dx dy}{\frac{1}{2} (\lambda_n^2 + \lambda_m^2) \left(\frac{1}{2} (\lambda_n^2 + H_3^2) \left(L + \frac{H_4}{\lambda_n^2 + H_4^2} \right) + \frac{H_3}{2} \right) \left(\frac{1}{2} (\lambda_m^2 + H_1^2) \left(M + \frac{H_2}{\lambda_m^2 + H_2^2} \right) + \frac{H_1}{2} \right)}$$

Where λ_n are positive roots of: $(H_3 H_4 - \lambda_n^2) \sin \lambda_n L + (H_3 + H_4) \lambda_n \cos \lambda_n L = 0$

Where λ_m are positive roots of: $(H_1 H_2 - \lambda_m^2) \sin \lambda_m M + (H_1 + H_2) \lambda_m \cos \lambda_m M = 0$

Solution of BVP for Poisson's Equation (superposition principle):

$$u(x, y) = u_5(x, y) + u_6(x, y)$$