first order o.d.e.

LINEAR O.D.E.

$$y' + P(x)y = Q(x)$$

integrating factor $\mu(x) = e^{\int P(x)dx}$

general solution $y(x) = c\mu^{-1}(x) + \mu^{-1}(x)\int \mu(x)Q(x)dx$

Initial Value Problem $y(x_0) = y_0$

solution $y(x) = y_0\mu(x_0)\mu^{-1}(x) + \mu^{-1}(x)\int_{x_0}^{x} \mu(x)Q(x)dx$

LINEAR O.D.E. WITH CONSTANT COEFFICIENTS

$$y' + ay = Q(x)$$

integrating factor $\mu(x) = e^{ax}$

general solution $y(x) = ce^{-ax} + e^{-ax}\int e^{ax}Q(x)dx$

Initial Value Problem $y(x_0) = y_0$

solution $y(x) = y_0 e^{a(x_0 - x)} + e^{-ax}\int_{x_0}^{x} e^{ax}Q(x)dx$

NONLINEAR EQUATIONS

$$M(x,y)dx + N(x,y)dy = 0$$

standard differential form

EXACT

test for exactness: if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then

there exists function $f(x,y)$ such that $df(x,y) = M(x,y)dx + N(x,y)dy$

general solution

$$f(x,y) = \int M dx + \int\left[N - \frac{\partial}{\partial y}\int M dx\right]dy = c$$

or

$$f(x,y) = \int N dy + \int\left[M - \frac{\partial}{\partial x}\int N dy\right]dx = c$$

reduce to exact ⇓ replace M by μM and N by μN

INTEGRATING FACTOR

equation becomes exact when multiplied by intgrating factor μ

a) if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ ⇒ integrating factor $\mu(x) = e^{\int f(x)dx}$

b) if $\dfrac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M} = g(y)$ ⇒ integrating factor $\mu(y) = e^{\int g(y)dy}$

SEPARABLE

equation is separable if it can be represented by $M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0$

then variables can be separated $f(x)dx + g(y)dy = 0$

general solution

$$\int f(x)dx + \int g(y)dy = c$$

check for suppressed solutions

⇑ reduce to separable

HOMOGENEOUS

equation is homogeneous if coefficients are homogeneous functions of the same order

$M(\lambda x, \lambda y) = \lambda^n M(x,y)$

$N(\lambda x, \lambda y) = \lambda^n N(x,y)$

homogeneous equation can be reduced to separable by:

a) change of variable $y = ux \quad dy = udx + xdu$ or $x = vy \quad dx = vdy + ydv$

b) conversion to polar coordinates $x = r\cos\theta \quad y = r\sin\theta$

$dx = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

homogeneous equation can be reduced to exact by integrating factor: $\mu = \dfrac{1}{N_1(x)M_2(y)}$

SPECIAL EQUATIONS

BERNOULLI

$$y' + P(x)y = Q(x)y^n$$

change of variable $y = z^{\frac{1}{1-n}}$ reduces equation to

linear o.d.e. $z' + (1-n)P(x)z = (1-n)Q(x)$

back substitution $z = y^{1-n}$

$y = 0$ is also a solution for n>0

RICATTI

$$y' = P(x)y^2 + Q(x)y + R(x)$$

if one particular solution is known u_1 then

change of variable $y = u_1 + \frac{1}{z}$ leads to

linear o.d.e. $z' + (2Pu_1 + Q)z = -P$

change of variable $y = -\frac{w'}{Pw}$ leads to

linear 2nd order o.d.e. $w'' - \left(\frac{P'}{P} + Q\right)w' + RPw = 0$

CLAIRAUT

$$y = xy' + f(y')$$

set of solutions $y = cx + f(c)$

and one parametric solution

$x = -f'(t)$

$y = f(t) - tf'(t)$

APPROXIMATE METHODS

IVP

$y' = f(x,y)$

$y(x_0) = y_0$

PICARD'S METHOD

choose starting approximation $y_{(0)}(x)$ then

iterate $y_{(j+1)}(x) = y_0 + \int_{x_0}^{x} f(x, y_{(j)}(x))dx$

where $j=0,1,2,\ldots$ is the number of iteration

TAYLOR SERIES

$$y(x) = y(x_0) + \frac{y'(x_0)}{1}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \ldots$$

$y(x_0) = y_0$

$y'(x_0) = f(x_0, y_0)$

$y''(x_0) = \frac{\partial}{\partial x}f(x_0, y_0)$

. . .