bessel functions

bessel equation (BE)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

BE can be obtained from eqn $r^2R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0$

by the change of variable $y(x) = R(r)$ $x = \lambda r$

solutions of BE are

bessel function of the 1st kind of order ν

when $\nu \neq$ *integer*

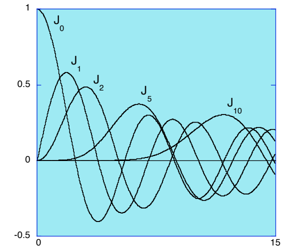
$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}$$

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)}$$

when $n = 1, 2, \ldots$ functions of integer order

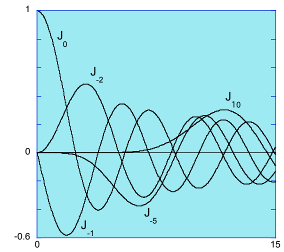
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!\,(n+k)!}$$

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,(-n+k)!}$$



when $\nu \neq$ *integer* $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent

general solution of BE $y(x) = c_1J_\nu(x) + c_2J_{-\nu}(x)$

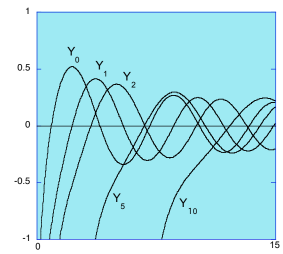


bessel function of the 2nd kind of order ν

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad Y_n(x) = \lim_{\nu\to n} Y_\nu(x) \qquad Y_{-n}(x) = (-1)^n Y_n(x)$$

$J_\nu(x)$ and $Y_\nu(x)$ are always linearly independent (when ν is integer or not)

general solution of BE $y(x) = c_1J_\nu(x) + c_2Y_\nu(x)$



properties

(the same properties also hold for bessel functions of the 2nd kind Y)

$J_{-n}(x) = (-1)^n J_n(x)$	$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x)$	$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x)$	$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$	$\int x^\nu J_{\nu-1}(x)dx = x^\nu J_\nu(x)$
$J_{-n}(x) = J_n(-x)$	$J_{\nu-1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu+1}(x)$	$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x}J_\nu(x)$	$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$	$\int x^{-\nu} J_{\nu+1}(x)dx = -x^{-\nu} J_\nu(x)$

orthogonality

functions $y_\nu(\lambda_n x)$ are combinations of $J_\nu(\lambda_n x)$ and $Y_\nu(\lambda_n x)$

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the values of parameter λ (eigenvalues)

for which boundary-value problem (Sturm-Liouville problem)

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \qquad x \in (x_1, x_2)$$

$$a_1y(x_1) - b_1y'(x_1) = 0 \qquad a_1^2 + b_1^2 \neq 0$$

$$a_2y(x_2) - b_2y'(x_2) = 0 \qquad a_2^2 + b_2^2 \neq 0$$

has non-trivial solutions (eigenfunctions)

$y_\nu(\lambda_1x), y_\nu(\lambda_2x), y_\nu(\lambda_3x), \ldots$

then $\{y_\nu(\lambda_nx)\}$ $n = 1, 2, \ldots$

is a complete set of functions

orthogonal on (x_1, x_2) w.r.t. weight x

$$\int_{x_1}^{x_2} xy_\nu(\lambda_nx)y_\nu(\lambda_mx)dx = 0 \quad \text{when } n \neq m$$

bessel-fourier series

$$f(x) = \sum_{n=1}^{\infty} c_ny_\nu(\lambda_nx)$$

$$c_n = \frac{\int_{x_1}^{x_2} xy_\nu(\lambda_nx)f(x)dx}{\int_{x_1}^{x_2} xy_\nu^2(\lambda_nx)dx}$$

modified bessel equation (MBE)

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

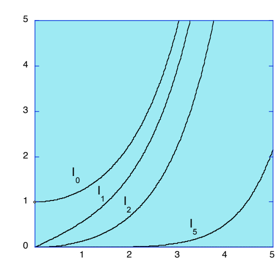
solutions of MBE are

modified bessel function of the 1st kind of order ν

$$I_\nu(x) = (-1)^{\nu}J_\nu(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)} \qquad I_{-\nu}(x) = (-1)^{-\nu}J_{-\nu}(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)}$$

when $\nu \neq$ *integer* $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly independent

general solution of MBE $y(x) = c_1I_\nu(x) + c_2I_{-\nu}(x)$



modified bessel function of the 2nd kind of order ν

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi} \qquad K_n(x) = \lim_{\nu\to n} K_\nu(x)$$

$I_\nu(x)$ and $K_\nu(x)$ are always linearly independent

general solution of MBE $y(x) = c_1I_n(x) + c_2K_n(x)$