TABLES

first order o.d.e.

LINEAR O.D.E.

$$y' + P(x)y = Q(x)$$

integrating factor $\mu(x) = e^{\int P(x)dx}$

general solution $y(x) = c\mu^{-1}(x) + \mu^{-1}(x)\int \mu(x)Q(x)dx$

Initial Value Problem $y(x_0) = y_0$

solution $y(x) = y_0\mu(x_0)\mu^{-1}(x) + \mu^{-1}(x)\int_{x_0}^{x}\mu(x)Q(x)dx$

LINEAR O.D.E. WITH CONSTANT COEFFICIENTS

$$y' + ay = Q(x)$$

integrating factor $\mu(x) = e^{ax}$

general solution $y(x) = ce^{-ax} + e^{-ax}\int e^{ax}Q(x)dx$

Initial Value Problem $y(x_0) = y_0$

solution $y(x) = y_0 e^{a(x_0 - x)} + e^{-ax}\int_{x_0}^{x} e^{ax}Q(x)dx$

NONLINEAR EQUATIONS

$$M(x,y)dx + N(x,y)dy = 0$$

standard differential form

EXACT

test for exactness: if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then

there exists function $f(x,y)$ such that $df(x,y) = M(x,y)dx + N(x,y)dy$

general solution

$$f(x,y) = \int Mdx + \int\left(N - \frac{\partial}{\partial y}\int Mdx\right)dy = c$$

or

$$f(x,y) = \int Ndy + \int\left(M - \frac{\partial}{\partial x}\int Ndy\right)dx = c$$

reduce to exact ⇓ replace M by μM and N by μN

INTEGRATING FACTOR

equation becomes exact when multiplied by intgrating factor μ

a) if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ ⇒ integrating factor $\mu(x) = e^{\int f(x)dx}$

b) if $\dfrac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M} = g(y)$ ⇒ integrating factor $\mu(y) = e^{\int g(y)dy}$

SEPARABLE

equation is separable if it can be represented by $M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0$

then variables can be separated $f(x)dx + g(y)dy = 0$

general solution

$$\int f(x)dx + \int g(y)dy = c$$

check for suppressed solutions

⇑ reduce to separable

HOMOGENEOUS

equation is homogeneous if coefficients are homogeneous functions of the same order

$M(\lambda x, \lambda y) = \lambda^n M(x,y)$

$N(\lambda x, \lambda y) = \lambda^n N(x,y)$

homogeneous equation can be reduced to separable by:

a) change of variable $y = ux$, $dy = udx + xdu$ or $x = vy$, $dx = vdy + ydv$

b) conversion to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$; $dx = \cos\theta dr - r\sin\theta d\theta$, $dy = \sin\theta dr + r\cos\theta d\theta$

homogeneous equation can be reduced to exact by integrating factor: $\mu = \dfrac{1}{N_1(x)M_2(y)}$

SPECIAL EQUATIONS

BERNOULLI

$$y' + P(x)y = Q(x)y^n$$

change of variable $y = z^{\frac{1}{1-n}}$ reduces equation to

linear o.d.e. $z' + (1-n)P(x)z = (1-n)Q(x)$

back substitution $z = y^{1-n}$

$y = 0$ is also a solution for n>0

RICATTI

$$y' = P(x)y^2 + Q(x)y + R(x)$$

if one particular solution is known u_1 then

change of variable $y = u_1 + \frac{1}{z}$ leads to

linear o.d.e. $z' + (2Pu_1 + Q)z = -P$

change of variable $y = -\frac{w'}{Pw}$ leads to

linear 2nd order o.d.e. $w'' - \left(\frac{P'}{P} + Q\right)w' + RPw = 0$

CLAIRAUT

$$y = xy' + f(y')$$

set of solutions $y = cx + f(c)$

and one parametric solution $x = -f'(t)$, $y = f(t) - tf'(t)$

APPROXIMATE METHODS

IVP

$y' = f(x,y)$

$y(x_0) = y_0$

PICARD'S METHOD

choose starting approximation $y_{(0)}(x)$ then

iterate $y_{(j+1)}(x) = y_0 + \int_{x_0}^{x} f(x, y_{(j)}(x))dx$

where $j = 0,1,2,\ldots$ is the number of iteration

TAYLOR SERIES

$$y(x) = y(x_0) + \frac{y'(x_0)}{1}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \ldots$$

$y(x_0) = y_0$

$y'(x_0) = f(x_0, y_0)$

$y''(x_0) = \frac{\partial}{\partial x} f(x_0, y_0)$

. . .

linear o.d.e.

LINEAR O.D.E.

$a_i(x), f(x) \in C(D)$ $\quad D \subset R$

nth ORDER

$$L_n y \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

linear o.d.e. is normal in D if $a_0(x) \neq 0$ for all $x \in D$

initial value problem

$$L_n y = f(x)$$
$$y(x_0) = k_1 \quad y'(x_0) = k_2 \quad \cdots \quad y^{(n-1)}(x_0) = k_n$$

Theorem
if equation is normal, then IVP has the unique solution

FUNDAMENTAL SET

any set of n linearly independent solutions $y_1(x), \ldots, y_n(x)$

of homogeneous linear o.d.e. $L_n y = 0$

is said to be a fundamental set (basis functions)

COMPLIMENTARY SOLUTION

complimentary solution (complete solution) of homogeneous o.d.e. $L_n y = 0$

is given by a linear combinatnation of basis functions

$$y_c(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \qquad c_i \in R$$

complimentary solution is a vector space spanned by basis functions

COMPLETE SOLUTION

complete solution of non-homogeneous linear o.d.e. $L_n y = f(x)$

consists of complimentary solution (complete solution of homogeneous equation) and any particular solution of non-homogeneous equation

$$y(x) = c_1 y_1(x) + \ldots + c_n y_n(x) + y_p(x)$$

HOMOGENEOUS LINEAR O.D.E. WITH CONSTANT COEFFICIENTS

nth ORDER $\quad L_n y = 0$

looking for solution in the form $y = e^{mx}$

substitution into equation yields an auxilary equation which has n roots (real or complex)

auxilary equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_m = 0$$

fundamental set includes

case (1) real root of multiplicity k $\quad m \quad$ $e^{mx}, \; xe^{mx}, \; \ldots, \; x^{k-1}e^{mx}$

case (2) conjugate pair complex roots $\quad m_1 = a + ib, \; m_2 = a - ib \quad$ $e^{ax}\cos bx$ and $e^{ax}\sin bx$

2nd ORDER

$$a_0 y'' + a_1 y' + a_2 = 0$$

auxialary equation $a_0 m^2 + a_1 m + a_2 = 0$ $\quad$ roots $m_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$

complete solution

case (1) real roots $m_1 \neq m_2$ $\qquad y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

case (2) repeated real root $m_1 = m_2 = m = \dfrac{-a_1}{2a_0}$ $\qquad y = c_1 e^{mx} + c_2 x e^{mx}$

case (3) conjugate pair complex roots $m_1 = a + ib$, $m_2 = a - ib$ $\qquad y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

REDUCTION OF ORDER

2nd ORDER

if y_1 is a particular solution of homogeneous equation $L_n y = 0$

then second linear independent solution can be found by formula

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{y_1^2} dx$$

WRONSKIAN

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem

$\{y_1, \ldots, y_n\}$ are linearly independent in D

if $W(y_1, \ldots, y_n) \neq 0$ for all $x \in D$

PARTICULAR SOLUTION OF NON-HOMOGENEOUS O.D.E.

(I) variation of parameter (Lagrange's method)

2nd ORDER $\quad L_2 y = f(x)$ $\quad$ looking for solution in the form $y_p = u_1 y_1 + u_2 y_2$

where $\{y_1, y_2\}$ is a fundamental set

unknown functions are determined by equations

$$u_1 = -\int \frac{y_2}{W(y_1, y_2)} \frac{f(x)}{a_0(x)} dx \qquad u_2 = \int \frac{y_1}{W(y_1, y_2)} \frac{f(x)}{a_0(x)} dx$$

nth ORDER $\quad L_n y = f(x)$ $\quad$ looking for solution in the form $y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$

where $\{y_1, \ldots, y_n\}$ is a fundamental set

unknown functions are determined by equations

$$u_m = (-1)^m \int \frac{W(y_1, \ldots, y_{m-1}, y_{m+1}, \ldots, y_n)}{W(y_1, \ldots, y_n)} \frac{f(x)}{a_0(x)} dx \qquad m = 1, \ldots, n$$

(II) method of undetermined coefficients

nth ORDER $\quad L_n y = f(x)$ $\quad$ where $f(x) = e^{ax}\left[p_i(x)\cos bx + q_j(x)\sin bx\right]$

1) if $a \pm ib$ is not a root of auxilary equation $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_m = 0$

then looking for solution in the form $y_p = e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right]$ $\quad k = \max\{i, j\}$

2) if $a \pm ib$ is a root of auxilary equation of multiplicity s

then looking for solution in the form $y_p = x^s e^{ax}\left[P_k(x)\cos bx + Q_k(x)\sin bx\right]$ $\quad k = \max\{i, j\}$

where

$$P_k(x) = A_0 x^k + A_1 x^{k-1} + \cdots + A_{k-1} x + A_k$$
$$Q_k(x) = B_0 x^k + B_1 x^{k-1} + \cdots + B_{k-1} x + B_k$$

are polynomials with unknown coefficients

which are found by substitution of trial y_p into equation $L_n y = f(x)$

EULER-CAUCHY EQUATION

(I)

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = f(x)$$

change of variable $x = e^z \quad z = \ln x$

yields a linear o.d.e. with constant coefficients

1st ORDER $\quad a_0 y' + a_1 y = f(z)$

2nd ORDER $\quad a_0 y'' + (a_1 - a_0) y' + a_2 y = f(z)$

3rd ORDER $\quad a_0 y''' + (a_1 - 3a_0) y'' + (2a_0 - a_1 + a_2) y' + a_3 y = f(z)$

(II) homogeneous equation

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = 0$$

looking for solution in the form $y = x^m \quad y' = m x^{m-1} \quad y'' = m(m-1)x^{m-2}$

auxiarly equation $a_0 m(m-1)\cdots(m-n+1) + a_1 m(m-1)\cdots(m-n+2) + \cdots + a_n = 0$

2nd ORDER $\quad a_0 m^2 + (a_1 - a_0)m + a_2 = 0$

independent solutions / complete solution

case (1) $m_1 \neq m_2$ $\quad y_1 = |x|^{m_1}$, $y_2 = |x|^{m_2}$ $\qquad y = c_1 |x|^{m_1} + c_2 |x|^{m_2}$

case (2) $m_1 = m_2 = m$ $\quad y_1 = |x|^m$, $y_2 = |x|^m \ln|x|$ $\qquad y = c_1 |x|^m + c_2 |x|^m \ln|x|$

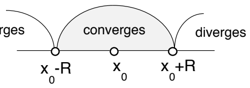
case (3) $m_1 = a + ib$, $m_2 = a - ib$ $\quad y_1 = |x|^a \cos(b\ln|x|)$, $y_2 = |x|^a \sin(b\ln|x|)$ $\qquad y = \left[c_1 \cos(b\ln|x|) + c_2 \sin(b\ln|x|)\right]|x|^a$

power-series solution

POWER SERIES

$$y(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

radius of convergence R



$$R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

TAYLOR SERIES

$$y(x)=y(x_0)+\frac{y'(x_0)}{1}(x-x_0)+\frac{y''(x_0)}{2!}(x-x_0)^2+\dots=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$$

MACLAUREN SERIES

$$y(x)=y(0)+\frac{y'(0)}{1}x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\dots=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n$$

ANALYTIC FUNCTION

function f(x) is called analytic at x_0 if it can be represented by Taylor series about x_0

BINOMIAL EXPANSION

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$$

binomial coefficients

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

THEOREM 2.6 (identity theorem)

a) $\sum_{n=0}^{\infty}a_nx^n=0$ for all x if and only if $a_n=0$

b) $\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}b_nx^n$ for all x if and only if $a_n=b_n$

ALGEBRAIC OPERATIONS

$$f(x)=\sum_{n=0}^{\infty}a_nx^n \qquad g(x)=\sum_{n=0}^{\infty}b_nx^n$$

sum of two power series

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

product of two power series (Cauchy product)

$$f(x)g(x)=\left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n=\sum_{n=0}^{\infty}c_nx^n$$

$\frac{1}{1-x}=1+x+x^2+\dots=\sum_{n=0}^{\infty}x^n \qquad |x|<1$

$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n \qquad -1<x<1$

$\frac{1}{x}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n \qquad 0<x<2$

$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!} \qquad |x|<\infty$

$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \qquad |x|<\infty$

$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \qquad |x|<\infty$

$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \qquad |x|<\infty$

$(1+x)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n \qquad |x|<1$

$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \qquad |x|<1$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad e^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^nx^n}{n!} \qquad |x|<\infty$

$a^x=\sum_{n=0}^{\infty}\frac{(\ln a)^nx^n}{n!} \qquad |x|<\infty$

$\ln x=\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{n} \qquad 0<x\le 2$

$\ln x=\sum_{n=0}^{\infty}\frac{2}{(2n+1)}\left(\frac{x-1}{x+1}\right)^{2n+1} \qquad x>0$

$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \qquad |x|<1$

HOMOGENEOUS LINEAR SECOND ORDER O.D.E.

$$(*)\qquad a_0(x)y''+a_1(x)y'+a_2(x)y=0$$

initial conditions:

$y(x_0)=y_0$

$y'(x_0)=y_1$

TAYLOR SERIES SOLUTION

determine values of function y(x) and its derivatives at the point of expansion x_0:

from initial conditions

$y(x_0)=y_0$

$y'(x_0)=y_1$

from equation

$$y''(x_0)=-\frac{a_1(x_0)}{a_0(x_0)}y'(x_0)-\frac{a_2(x_0)}{a_0(x_0)}y(x_0)$$

differentiate consequently original equation and substitute $x=x_0$

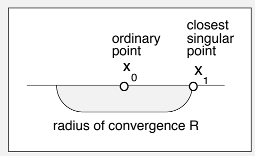
. . .

definition

x_0 is ordinary point if $a_0(x_0)\neq 0$

x_0 is singular point if $a_0(x_0)=0$

THEOREM 2.10 (power-series solution about ordinary point)



Let $a_k(x)$ be analytic, then

two linearly independent solutions of Eq.(∗) can be found as a power series about ordinary point x_0:

1) assume: $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$

2) calculate derivatives: $y'(x)=\sum_{n=1}^{\infty}nc_n(x-x_0)^{n-1}$

$y''(x)=\sum_{n=2}^{\infty}n(n-1)c_n(x-x_0)^{n-2}$

3) substitute into Eq. (∗)

4) comparison of coefficients (identity theorem) yeilds the recursive equation for coefficients c_n

c_0, c_1 can be taken arbitrary (or as parameters for complete solution)

standard form of Eq. (∗) $\qquad y''+p(x)y'+q(x)y=0 \qquad (**)$

definition 2.2 singular point x_0=0 is regular if

$xp(x)=p_0+p_1x+p_2x^2+\cdots$

$x^2q(x)=q_0+q_1x+q_2x^2+\cdots$

are analytic at x=0

indicial equation $r^2+(p_0-1)r+q_0=0$ roots $r_{1,2}=\frac{1-p_0\pm\sqrt{(1-p_0)^2-4q_0}}{2}$

THEOREM 2.11 (Frobenius method for regular singular point)

Let x_0=0 be a regular singular point of Eq. (∗∗) and r_1 and r_2 be the roots of indicial equation, then two linearly independent solutions y_1 and y_2 of Eq. (∗∗) can be found in the form: (for 0<|x|<R)

case 1	r_1 - r_2 is not integer	$y_1=\sum_{n=0}^{\infty}c_nx^{n+r_1}$, $c_0\neq 0$	$y_2=\sum_{n=0}^{\infty}d_nx^{n+r_2}$, $d_0\neq 0$
case 2	r_1 - r_2 is positive integer	$y_1=\sum_{n=0}^{\infty}c_nx^{n+r_1}$, $c_0\neq 0$	$y_2=\sum_{n=0}^{\infty}d_nx^{n+r_2}+cy_1\ln\lvert x\rvert$, $d_0\neq 0$
case 3	$r_1=r_2=r$	$y_1=\sum_{n=0}^{\infty}c_nx^{n+r}$, $c_0\neq 0$	$y_2=\sum_{n=1}^{\infty}d_nx^{n+r}+y_1\ln\lvert x\rvert$, all d_n can be zero

linear systems of first order o.d.e.

STANDARD FORM

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n + f_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n + f_2 \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n + f_n \end{aligned}$$

$a_{ij}(t)$, $f_i(t)$ i,j=1,..,n $t \in I \subset R$ are continuous functions

MATRIX FORM

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

SOLUTION VECTOR

is a column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ which satisfies the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$

HOMOGENEOUS SYSTEMS

$$\mathbf{x}' = A\mathbf{x}$$

COMPLETE SOLUTION set of all solutions to homogeneous system is a vector space of dimension n

FUNDAMENTAL SET any set of *n* linearly independent solution vectors

$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ (a basis for solution space)

FUNDAMENTAL MATRIX columns are vectors from fundamental set

$$X = [\mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \cdots \ \ \mathbf{x}_n]$$

wronskian $W = \det X(t) \neq 0 \quad \forall \ t \in I$

COMPLETE SOLUTION (complementary function)

$$\mathbf{x}_c = X\mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is a vector of coefficients

INITIAL-VALUE PROBLEM

$$\mathbf{x}' = A\mathbf{x} \qquad \mathbf{x}(t_0) = \mathbf{k}$$

SOLUTION OF IVP

$$\mathbf{x} = X(t)X^{-1}(t_0)\mathbf{k}$$

NON-HOMOGENEOUS SYSTEMS

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

COMPLETE SOLUTION

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$$

where $\mathbf{x}_p$ is any particular solution of non-homogeneous system

VARIATION OF PARAMETER

$$\mathbf{x}_p = X(t)\int X^{-1}(t)\mathbf{f}(t)dt$$

COMPLETE SOLUTION

$$\mathbf{x} = X(t)\mathbf{c} + X(t)\int X^{-1}(t)\mathbf{f}(t)dt$$

INITIAL-VALUE PROBLEM

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \qquad \mathbf{x}(t_0) = \mathbf{k}$$

SOLUTION OF IVP

$$\mathbf{x} = X(t)X^{-1}(t_0)\mathbf{k} + X(t)\int_{t_0}^{t} X^{-1}(s)\mathbf{f}(s)ds$$

FUNDAMENTAL SET OF SYSTEM WITH CONSTANT COEFFICIENTS

$$\mathbf{x}' = A\mathbf{x}$$

A is an $n \times n$ matrix with real constant coefficients

looking for solution in the form

$$\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = \mathbf{K}e^{\lambda t}$$

EIGENVALUE PROBLEM

egenvalues λ are the roots of algebraic characteristic equation $\det(A - \lambda I) = 0$

eigenvector $\mathbf{K}$ corresponding to λ is non-trivial solution of equation $(A - \lambda I)\mathbf{K} = \mathbf{0}$

① DISTINCT REAL EIGENVALUES $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{K}_1, \dots, \mathbf{K}_m$

FUNDAMENTAL SET

$$\mathbf{K}_1 e^{\lambda_1 t}, \quad \mathbf{K}_2 e^{\lambda_2 t} \quad , \dots, \quad \mathbf{K}_n e^{\lambda_n t}$$

② REPEATED EIGENVALUES λ is eigenvector of multiplicity m

case 1 there are linearly independent eigenvectors $\mathbf{K}_1, \dots, \mathbf{K}_m$ corresponding to λ

FUNDAMENTAL SET includes

$$\mathbf{K}_1 e^{\lambda t}, \quad \mathbf{K}_2 e^{\lambda t} \quad , \dots, \quad \mathbf{K}_m e^{\lambda t}$$

case 2 there is only one eigenvector $\mathbf{K}$ corresponding to λ

find vectors $\mathbf{K}, \mathbf{P}, \mathbf{Q}, \dots$ solutions of the matrix equations

$$(A - \lambda I)\mathbf{K} = \mathbf{0}$$

$$(A - \lambda I)\mathbf{P} = \mathbf{K}$$

$$(A - \lambda I)\mathbf{Q} = \mathbf{P}$$

FUNDAMENTAL SET includes

$$\mathbf{K}e^{\lambda t}, \quad \mathbf{K}te^{\lambda t} + \mathbf{P}e^{\lambda t}, \quad \mathbf{K}\frac{t^2}{2}e^{\lambda t} + \mathbf{P}te^{\lambda t} + \mathbf{Q}e^{\lambda t} \quad \dots$$

③ COMPLEX EIGENVALUES $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ corresponding eigenvectors $\mathbf{K}_1$, $\overline{\mathbf{K}}_1$

solution vectors $\mathbf{x}_1 = \mathbf{K}_1 e^{\lambda_1 t}$ $\quad \mathbf{x}_2 = \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}$

FUNDAMENTAL SET includes real form of solution vectors

$$\mathbf{x}_1 = (\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t)e^{\alpha t}$$

$$\mathbf{x}_2 = (\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t)e^{\alpha t}$$

where $\mathbf{K}_1 = \mathrm{Re}(\mathbf{K}_1) + i\,\mathrm{Im}(\mathbf{K}_1) = \mathbf{B}_1 + i\mathbf{B}_2$

MATRIX EXPONENTIAL

is a matrix represented by a power series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(t\mathbf{A})^k}{k!} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots$$

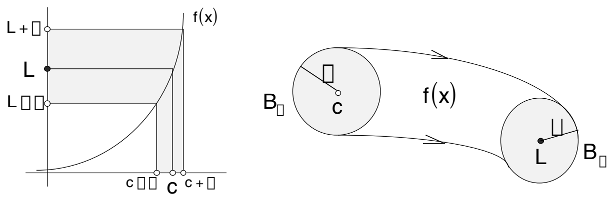
general solution $\mathbf{x} = e^{t\mathbf{A}}\mathbf{c} + e^{t\mathbf{A}}\int e^{-t\mathbf{A}}\mathbf{f}(t)dt$

solution of IVP $\mathbf{x} = e^{(t-t_0)\mathbf{A}}\mathbf{k} + e^{t\mathbf{A}}\int_{t_0}^{t} e^{-s\mathbf{A}}\mathbf{f}(s)ds$

limits

definition of limit

$\lim_{x\to c} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$



uniqueness

if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$ then $L = M$

L'Hospital's Rule

(indeterminate $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

B_c = open neighborhood of point c

$f(x), g(x)$ are differentiable on B_c

Let $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \infty$, $\lim_{x\to c} g(x) = \infty$

then

if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L$

if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = \infty \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \infty$

algebra of limits

in the following statements we assume that the given limits $\lim_{x\to c} f(x)$ exist where c can be c, c^+, c^-, ∞, or $-\infty$

$\lim k = k$

$\lim k f(x) = k \lim f(x)$

$\lim[f(x) + g(x)] = \lim f(x) + \lim g(x)$

$\lim[f(x) - g(x)] = \lim f(x) - \lim g(x)$

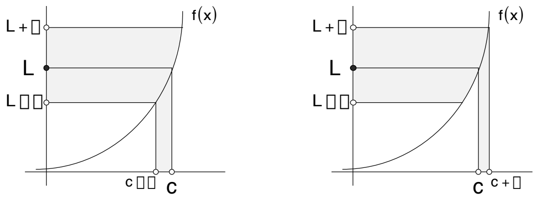
$\lim f(x) g(x) = \lim f(x) \lim g(x)$

$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ if $\lim g(x) \neq 0$

if $\lim f(x) \neq 0$ and $\lim g(x) = 0$ then $\frac{\lim f(x)}{\lim g(x)}$ DNE

$\lim f(x) = L \iff \lim[f(x) - L] = 0$

one-sided limits



$\lim_{x\to c^-} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $c - \delta < x < c$ then $|f(x) - L| < \varepsilon$

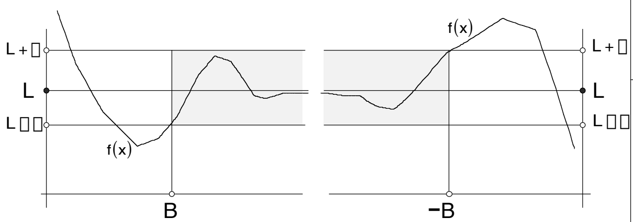
$\lim_{x\to c^+} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $c < x < c + \delta$ then $|f(x) - L| < \varepsilon$

theorem

$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$

limits at infinity

(horizontal assymtotes)

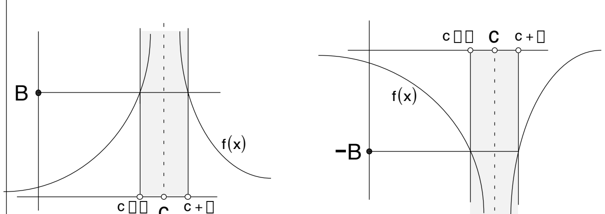


$\lim_{x\to \infty} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $B > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > B$

$\lim_{x\to -\infty} f(x) = L \iff$ for any $\varepsilon > 0$ there exists $B > 0$ such that $|f(x) - L| < \varepsilon$ for all $x < -B$

infinite limits

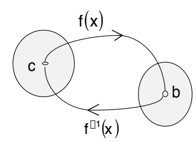
(vertical assymtotes)



$\lim_{x\to c} f(x) = \infty \iff$ for any $B > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > B$

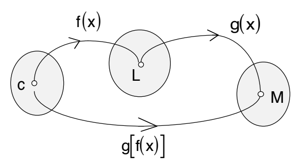
$\lim_{x\to c} f(x) = -\infty \iff$ for any $B > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) < -B$

inverse function



$\lim_{x\to c} f(x) = b \iff \lim_{x\to b} f^{-1}(x) = c$

composite function

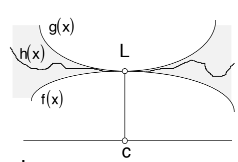


$\lim_{x\to c} f(x) = L$, $\lim_{x\to L} g(x) = M$ then $\lim_{x\to c} g[f(x)] = M$

comparison

$\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, $f(x) \le g(x)$ then $L \le M$

squizze play



$\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = L$, $f(x) \le h(x) \le g(x)$ then $\lim_{x\to c} h(x) = L$

indeterminates and reduction to L'Hospital's rule

1) $f \cdot g \sim 0 \cdot \infty \Rightarrow f \cdot g = \frac{f}{\frac{1}{g}} \sim \frac{0}{0}$

$\Rightarrow f \cdot g = \frac{g}{\frac{1}{f}} \sim \frac{\infty}{\infty}$

2) $f^g \sim 0^0$, $f^g \sim \infty^0$, $f^g \sim 1^\infty$ $\Rightarrow f^g = e^{g \ln f}$

3) $f - g \sim \infty - \infty$

a) multiply and devide by conjugate

$$\frac{(f - g)(f + g)}{f + g}$$

b) apply L'Hospital's rule to

$$f - g = \frac{\frac{1}{g} - \frac{1}{f}}{\frac{1}{f \cdot g}}$$

remarkable limits

$\lim_{x\to c} x = c$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} x^x = 1$

$\lim_{x\to 0} x \ln x = 0$

$\lim_{x\to 0} (1 + ax)^{1/x} = e^a$

$\lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

$\lim_{x\to \infty} x^{\frac{1}{x}} = 1$

$\lim_{x\to \infty} \frac{\ln x}{x} = 0$

rational function

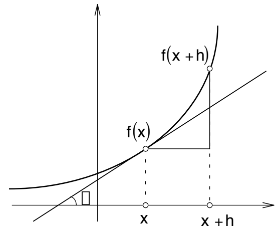
$$\lim_{x\to \infty} \frac{p_n x^n + \ldots + p_1 x + p_0}{q_m x^m + \ldots + q_1 x + q_0} = \begin{cases} 0 & \text{if } n < m \\ \frac{p_n}{q_n} & \text{if } n = m \\ \pm\infty & \text{if } n > m \end{cases}$$

differentiation

derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if limit exists then
function f(x) is differentiable and
f'(x) is a derivative of function f(x)



equation of tangent line $y = f'(x_0)(x - x_0) + f(x_0)$

$f'(x) = m$ slope of tangent line

$= \tan\theta$

differentiation rules

$(cu)' = cu'$

$(u+v)' = u' + v'$

$(u-v)' = u' - v'$

$(uv)' = u'v + uv'$

$(uvw)' = u'vw + uv'w + uvw'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\left(\frac{1}{v}\right)' = \frac{-v'}{v^2}$

chain rule

$y = f[u(x)]$

$$\frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$$

example: $y = \sin(x^2)$ $u = x^2$

$$y' = \frac{d(\sin u)}{du}\frac{du}{dx} = (\cos u)(2x) = 2x\cos(x^2)$$

derivative of inverse function

$y = f(x)$

$y = f^{-1}(x)$

$$\left[f^{-1}(x)\right]' = \frac{1}{f'\left[f^{-1}(x)\right]}$$

1) find $f'(x)$

2) write $\frac{1}{f'(x)}$

3) replace x by $f^{-1}(x)$ and simplify expression

example:

$f(x) = e^x$

$f^{-1}(x) = \ln x$

1) find $f'(x) = e^x$

2) write $\frac{1}{f'(x)} = \frac{1}{e^x}$

3) replace x by and simplify expression

$$(\ln x)' = \left[f^{-1}(x)\right]' = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

implicit differentiation

given implicit function $f(x,y) = 0$

find y'

differentiate using differentiation rules

terms with x only differetiate as function of x

terms with y differentiate with the chain rule $\frac{d}{dx}g(y) = \frac{dg}{dy}y'$

solve for y'

example:

$f(x,y) = xy + \sin y - 1 = 0$

$(xy)' + (\sin y)' - (1)' = 0$

$y + xy' + (\cos y)y' = 0$

$y' = \frac{-y}{x + \cos y}$

table of derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin x$	$\cos x$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
x	1	$\cos x$	$-\sin x$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
x^n	nx^{n-1}	$\tan x$	$1+\tan^2 x$ $\frac{1}{\cos^2 x}$ $\sec^2 x$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
e^x	e^x	$\cot x$	$-1-\cot^2 x$ $\frac{-1}{\sin^2 x}$ $-\csc^2 x$	$\cot^{-1} x$	$\frac{-1}{1+x^2}$
a^x	$a^x \ln a$			$\csc^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\ln x$	$\frac{1}{x}$	$\csc x$	$-\csc x \cot x$	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\log_a x$	$\frac{1}{x \ln a}$	$\sec x$	$\sec x \tan x$		

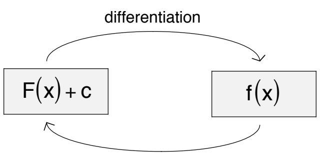
integration

indefinite integral

$F(x)$ is **antiderivative** of $f(x)$ if $F'(x) = f(x)$

if $F(x)$ is **antiderivative** of $f(x)$ then

$F(x) + c$ is also antiderivative of $f(x)$



indefinite integral (operation to find antiderivative)

$$\int f(x)dx = F(x) + c$$

table of antiderivatives

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1}$ $n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{e^{ax}}{a}$
a^x	$\frac{a^x}{\ln a}$
$\ln x$	$x\ln x - x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

u - substitution

$$\int f[u(x)]u'(x)dx = \int f(u)du$$

$$\int_a^b f[u(x)]u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

integration by parts

$$\int udv = uv - \int vdu$$

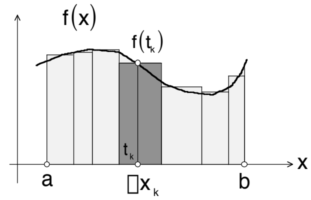
$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

definite integral

partition $\Delta = \{a = x_0, x_1, ..., x_k, ..., x_{n-1}, x_n = b\}$

norm of partition $|\Delta| = \max_k \Delta x_k$

t_k is an arbitrary point in the subinterval $[x_{k-1}, x_k]$ $\Delta x_k = x_k - x_{k-1}$



definite integral

$$\int_a^b f(x)dx = \lim_{|\Delta| \to 0} \sum_{k=1}^{n} f(t_k)\Delta x_k$$

if the limit of Riemann's sum exists

function $f(x)$ is said integrable on $[a,b]$

Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

Area

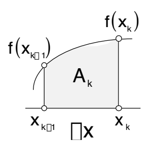
if $f(x) \geq 0$ for all $x \in [a,b]$ then

definite integral is interpreted as the area under $f(x)$ over $[a,b]$

if $f(x) \leq 0$ for all $x \in [a,b]$ then

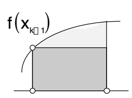
definite integral is interpreted as the negative area between $f(x)$ and x-axis over $[a,b]$

Numerical approximation of definite integral



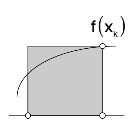
regular subdivision on n subintervals $\Delta x = \frac{b-a}{n}$ $x_k = a + k\Delta x$ $k = 0,1,...,n$

$$A = \int_a^b f(x)dx = A_1 + ... + A_n$$



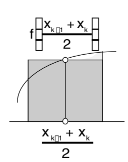
Left-Hand Sum

$$A \approx [f(x_0) + ... + f(x_{n-1})]\Delta x$$



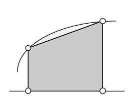
Right-Hand Sum

$$A \approx [f(x_1) + ... + f(x_n)]\Delta x$$



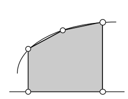
Mid-Point Sum

$$A \approx \left[f\left(\frac{x_0 + x_1}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Delta x$$



Trapezoidal Rule

$$A \approx [f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)]\frac{\Delta x}{2}$$



Simpson's Rule

$$A \approx \left\{f(x_0) + 2[f(x_1) + ... + f(x_{n-1})] + 4\left[f\left(\frac{x_0 + x_1}{2}\right) + ... + f\left(\frac{x_{n-1} + x_n}{2}\right)\right] + f(x_n)\right\}\frac{\Delta x}{6}$$

properties

linearity

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \qquad c \in R$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

limit rules

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

additivity

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad c \in [a,b]$$

differentiation

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = f[u(x)]\cdot u'(x) - f[v(x)]\cdot v'(x)$$

comparison

if $f(x) \geq 0$ on $x \in [a,b]$ then

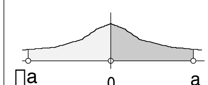
$$\int_a^b f(x)dx \geq 0$$

if $f(x) \geq g(x)$ on $x \in [a,b]$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

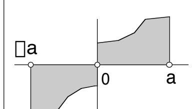
symmetric interval

$f(x)$ is even if $f(-x) = f(x)$



$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$f(x)$ is odd if $f(-x) = -f(x)$



$$\int_{-a}^{a} f(x)dx = 0$$

Average Value of Function $f(x)$ over $[a,b]$

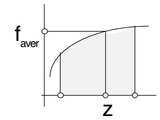
$$f_{aver} = \frac{\int_a^b f(x)dx}{b-a}$$

Mean Value Theorem

If function $f(x)$ is continuous on $[a,b]$

then there exists a point $z \in [a,b]$ such that

$$f(z) = f_{average}$$



F(x) is **antiderivative** of f(x) if	indefinite integral	definite integral	u - substitution	integration by parts	
$F'(x)=f(x)$	$\int f(x)dx = F(x)+c$	$\int_a^b f(x)dx = F(b)-F(a)$	$\int f[u(x)]u'(x)dx = \int f(u)du$	$\int u\,dv = uv - \int v\,du$	$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1}$ $\quad n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{e^{ax}}{a}$
a^x	$\frac{a^x}{\ln a}$
xe^{ax}	$\frac{e^{ax}}{a^2}(ax-1)$
$x^n e^{ax}$	$\frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx \quad n \geq 1$
$\frac{e^{ax}}{x}$	$\ln x + ax + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots$
$\ln x$	$x\ln x - x$
$x\ln x$	$\frac{x^2}{2}\ln x - \frac{x^2}{4}$
$x^n \ln x$	$x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right) \quad n \neq -1$
$\frac{(\ln x)^n}{x}$	$\frac{(\ln x)^{n+1}}{n+1} \quad n \neq -1$
$\frac{1}{x\ln x}$	$\ln\lvert\ln x\rvert$
$(\ln x)^2$	$x(\ln x)^2 - 2\ln x + 2x$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\ln\left(x+\sqrt{x^2 \pm a^2}\right)$
$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$

$f(x)$	$F(x)$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin x\cos x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin x\cos x}{2}$
$\tan^2 x$	$\tan x - x$
$\cot^2 x$	$-\cot x - x$
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\sin^n x$	$-\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$
$\cos^n x$	$\frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$
$\tan^n x$	$\frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$
$\cot^n x$	$-\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx$
$\sec^n x$	$\frac{\sec^{n-1}x\sin x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$
$\csc^n x$	$-\frac{\csc^{n-1}x\cos x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx$
$\sinh(ax)$	$\frac{\cosh(ax)}{a}$
$\cosh(ax)$	$\frac{\sinh(ax)}{a}$
$\tanh(ax)$	$\frac{\ln[\cosh(ax)]}{a}$
$\coth(ax)$	$\frac{\ln[\sinh(ax)]}{a}$

quadratic equation

$$ax^2+bx+c=0 \qquad x_{1,2} = \frac{-b \pm \sqrt{a^2-4bc}}{2a}$$

binomial formula

$$(x \pm y)^n = \sum_{k=0}^{n}\binom{n}{k}(\pm 1)^k x^{n-k}y^k \qquad \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$a^x = e^{x\ln a} \qquad \log_a x = \frac{\ln x}{\ln a}$$

hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh = \frac{e^x + e^{-x}}{2}$$

$$e^{a+ib} = e^a(\cos b + i\sin b)$$ Euler's formula

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\tan^2 x = \sec^2 x - 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x = \csc^2 x - 1 \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\sin x\cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

$$\sin x\sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos x\cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\sin(\theta+\phi) \qquad \phi = \tan^{-1}\frac{a}{b}$$

Liebniz' rule

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)}\frac{\partial f(x,t)}{\partial t}dx + f[b(t),t]\frac{db(t)}{dt} - f[a(t),t]\frac{da(t)}{dt}$$

convergence of infinite series

infinite series	$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$
partial sums	$s_n = a_0 + a_1 + a_2 + \cdots + a_n$

Definition infinite series is convergent if and only if the sequence of partial sums is convergent

$\sum_{k=0}^{\infty} a_k = L$ if and only if $s_k \to L$

test	condition		result
basic test	$a_k \underset{n\to\infty}{\not\to} 0$	$\Rightarrow$	$\sum a_k$ diverges
geometric	$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	$\Rightarrow$	$\lvert x\rvert < 1 \Rightarrow \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $\lvert x\rvert \ge 1 \Rightarrow$ diverges
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$	$\Rightarrow$	$p > 1 \Rightarrow$ converges $p \le 1 \Rightarrow$ diverges
telescoping	$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} [f(k) - f(k+1)]$ $= f(0) - f(n+1)$	$\Rightarrow$	$f(n+1) \to c \Rightarrow \sum_{k=0}^{\infty} a_k = f(0) - c$ converges $f(n+1) \to \pm\infty \Rightarrow \sum_{k=0}^{\infty} a_k$ diverges
absolute convergence test	$\sum \lvert a_k \rvert$ converges	$\Rightarrow$	$\sum a_k$ converges
alternating	$\sum_{k=0}^{\infty} (-1)^K a_k \quad a_k \ge 0$	$\Rightarrow$	$a_k \to 0$, $a_k > a_{k+1}$ $\Rightarrow \sum_{k=0}^{\infty} (-1)^K a_k$ converges

harmonic series $(p = 1)$

$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

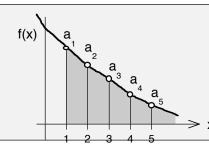
remainder

$\sum_{k=0}^{\infty} (-1)^K a_k = L \Rightarrow \lvert L - s_n \rvert < a_{n+1}$

series with non-negative terms $a_k \ge 0$

integral test

$\int_1^{\infty} f(x)dx$ converges $\Leftrightarrow \sum_{k=1}^{\infty} f(k)$ converges



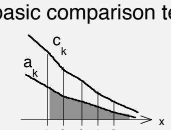
ratio test

if $\frac{a_{k+1}}{a_k} \to \rho$ then
- $\rho < 1 \Rightarrow \sum a_k$ converges
- $\rho = 1 \Rightarrow$ no conclusion
- $\rho > 1 \Rightarrow \sum a_k$ diverges

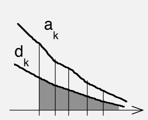
root test

if $(a_k)^{\frac{1}{k}} \to \rho$ then
- $\rho < 1 \Rightarrow \sum a_k$ converges
- $\rho = 1 \Rightarrow$ no conclusion
- $\rho > 1 \Rightarrow \sum a_k$ diverges

basic comparison test



$c_k \ge 0$, $c_k \ge a_k$, $\sum c_k$ converges $\Rightarrow \sum a_k$ converges



$d_k \ge 0$, $d_k \le a_k$, $\sum d_k$ diverges $\Rightarrow \sum a_k$ diverges

limit comparison test

$a_k \ge 0$, $b_k > 0$, $L > 0$ if $\frac{a_k}{b_k} \to L \Rightarrow$ both $\sum a_k, \sum b_k$ converge or both $\sum a_k, \sum b_k$ diverge

limits of sequences

$x^n \to 0 \quad \lvert x\rvert < 1$	$x^{\frac{1}{n}} \to 1 \quad x > 0$	$\frac{1}{n^{\alpha}} \to 0 \quad \alpha > 0$	$\frac{x^n}{n!} \to 0 \quad \forall x$	$\frac{\ln n}{n} \to 0$	$n^{\frac{1}{n}} \to 1$	$\left(1 + \frac{x}{n}\right)^n \to e^x$

fourier series

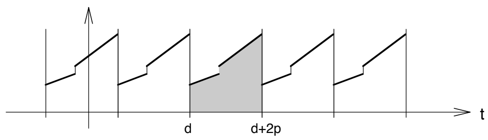
basic case $(d, d+2p)$

standard form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi}{p}t + b_n \sin\frac{n\pi}{p}t\right]$$

complex exponential forms

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{ni\pi}{p}t} = \sum_{n=-\infty}^{\infty} c_{-n} e^{\frac{-ni\pi}{p}t}$$



complex Fourier coefficients

$$c_n = \frac{1}{2p}\int_d^{d+2p} f(t) e^{\frac{-n\pi i}{p}t} dt \qquad c_{-n} = \frac{1}{2p}\int_d^{d+2p} f(t) e^{\frac{n\pi i}{p}t} dt$$

Fourier coefficients

$$a_0 = \frac{1}{p}\int_d^{d+2p} f(t)dt$$

$$a_n = \frac{1}{p}\int_d^{d+2p} f(t)\cos\frac{n\pi}{p} tdt$$

$$b_n = \frac{1}{p}\int_d^{d+2p} f(t)\sin\frac{n\pi}{p} tdt$$

relations

$$a_0 = 2c_0$$

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{a_n - ib_n}{2}$$

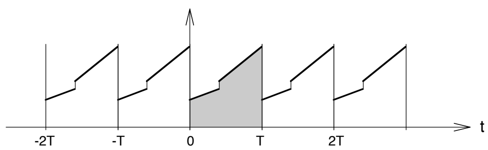
$$c_{-n} = \frac{a_n + ib_n}{2}$$

interval $(0, T)$

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n \cos\frac{2\pi n}{T}t + d_n \sin\frac{2\pi n}{T}t\right] = \sum_{n=-\infty}^{\infty} A_n e^{\frac{i2\pi n}{T}t} = c_0 + \sum_{n=1}^{\infty}\sqrt{c_n^2 + d_n^2}\,\sin\left(\frac{2\pi n}{T}t + \phi_n\right)$$

harmonic series

$$\tan\phi_n = \frac{c_n}{d_n}$$



$$c_0 = \frac{1}{T}\int_0^T f(t)dt$$

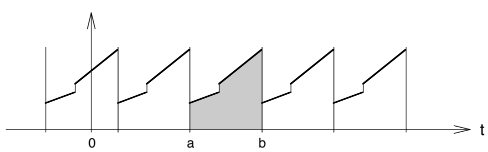
$$c_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n}{T} tdt$$

$$d_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi n}{T} tdt$$

$$A_n = \frac{1}{T}\int_0^T f(t) e^{\frac{-2ni\pi}{T}t} dt$$

arbitrary interval (a, b)

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n \cos\frac{2\pi n}{b-a}(t-a) + d_n \sin\frac{2\pi n}{b-a}(t-a)\right]$$



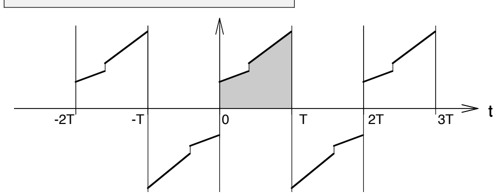
$$c_0 = \frac{1}{b-a}\int_a^b f(t)dt$$

$$c_n = \frac{2}{b-a}\int_a^b f(t)\cos\frac{2\pi n}{b-a}(t-a)dt$$

$$d_n = \frac{2}{b-a}\int_a^b f(t)\sin\frac{2\pi n}{b-a}(t-a)dt$$

sine series $(0, T)$

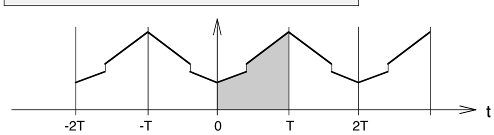
$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n}{T} t$$



$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{\pi n}{T} tdt$$

cosine series $(0, T)$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n}{T} t$$

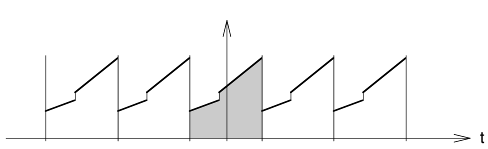


$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{\pi n}{T} tdt$$

symmetric interval $(-p, p)$

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left[c_n \cos\frac{\pi n}{p}t + d_n \sin\frac{\pi n}{p}t\right]$$

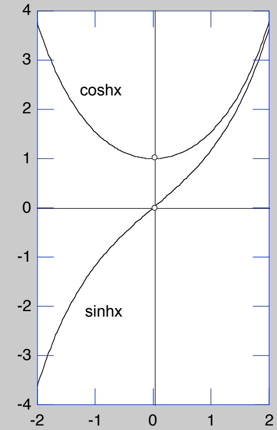


$$c_0 = \frac{1}{p}\int_{-p}^{p} f(t)dt$$

$$c_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{\pi n}{p} tdt$$

$$d_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{\pi n}{p} tdt$$

hyperbolic functions

definition	$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$	$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$	
derivative	$\sinh' x = \cosh x$	$\cosh' x = \sinh x$	
integration	$\int \sinh x dx = \cosh x$	$\int \cosh x dx = \sinh x$	
symmetry	$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$	
value at 0	$\sinh 0 = 0$	$\cosh 0 = 1$	
series expansion	$\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$	$\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$	
identities	$\cosh^2 x - \sinh^2 x = 1$	$\cosh x \pm \sinh x = e^{\pm x}$	

special equation

$$X''(x) - \lambda X(x) = 0$$

$\lambda =$ *constant*

auxilary equation $m^2 = \lambda$

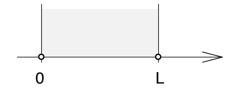
roots $m = \pm\sqrt{\lambda}$

general solution	derivative of general solution
$X(x) = \begin{cases} c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} & \lambda > 0 \\ c_1 + c_2 x & \lambda = 0 \\ c_1 \cos\sqrt{-\lambda}x + c_2 \sin\sqrt{-\lambda}x & \lambda < 0 \end{cases}$	$X'(x) = \begin{cases} c_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}x} & \lambda > 0 \\ c_2 & \lambda = 0 \\ -c_1\sqrt{-\lambda}\sin\sqrt{-\lambda}x + c_2\sqrt{-\lambda}\cos\sqrt{-\lambda}x & \lambda < 0 \end{cases}$
$X(x) = \begin{cases} c_1 \cosh\sqrt{\lambda}x + c_2 \sinh\sqrt{\lambda}x & \lambda > 0 \\ c_1 + c_2 x & \lambda = 0 \\ c_1 \cos\sqrt{-\lambda}x + c_2 \sin\sqrt{-\lambda}x & \lambda < 0 \end{cases}$	$X'(x) = \begin{cases} c_1\sqrt{\lambda}\sinh\sqrt{\lambda}x + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}x & \lambda > 0 \\ c_2 & \lambda = 0 \\ -c_1\sqrt{-\lambda}\sin\sqrt{-\lambda}x + c_2\sqrt{-\lambda}\cos\sqrt{-\lambda}x & \lambda < 0 \end{cases}$
$X(x) = \begin{cases} c_1 \cosh\sqrt{\lambda}(x - x_0) + c_2 \sinh\sqrt{\lambda}(x - x_0) & \lambda > 0 \\ c_1 + c_2(x - x_0) & \lambda = 0 \\ c_1 \cos\sqrt{-\lambda}(x - x_0) + c_2 \sin\sqrt{-\lambda}(x - x_0) & \lambda < 0 \end{cases}$	$X'(x) = \begin{cases} c_1\sqrt{\lambda}\sinh\sqrt{\lambda}(x - x_0) + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}(x - x_0) & \lambda > 0 \\ c_2 & \lambda = 0 \\ -c_1\sqrt{-\lambda}\sin\sqrt{-\lambda}(x - x_0) + c_2\sqrt{-\lambda}\cos\sqrt{-\lambda}(x - x_0) & \lambda < 0 \end{cases}$

sturm-liouville problem

$H_1 = \frac{h_1}{k_1} \qquad H_2 = \frac{h_2}{k_2}$

$X'' + \mu X = 0 \qquad X(x) \quad x \in [0, L]$



boundary conditions	eigenvalues $\mu_n = \lambda_n^2$	eigenfunctions X_n	norm $\lVert X_n \rVert^2 = \int_0^L X_n^2(x)\,dx$	kernel $K_n(x) = \frac{X_n(x)}{\lVert X_n \rVert}$
Dirichlet $X(0) = 0$ Dirichlet $X(L) = 0$	$\lambda_n = \frac{n\pi}{L}$ $\quad n = 1, 2, \ldots$	$\sin \frac{n\pi}{L} x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$
Neumann $X'(0) = 0$ Dirichlet $X(L) = 0$	$\lambda_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$ $\quad n = 0, 1, 2, \ldots$	$\cos\left(n + \frac{1}{2}\right)\frac{\pi}{L} x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}} \cos\left(n + \frac{1}{2}\right)\frac{\pi}{L} x$
Dirichlet $X(0) = 0$ Neumann $X'(L) = 0$	$\lambda_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$ $\quad n = 0, 1, 2, \ldots$	$\sin\left(n + \frac{1}{2}\right)\frac{\pi}{L} x$	$\frac{L}{2}$	$\sqrt{\frac{2}{L}} \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L} x$
Neumann $X'(0) = 0$ Neumann $X'(L) = 0$	$\lambda_n = \frac{n\pi}{L}$ $\quad n = 0, 1, 2, \ldots$	$\cos \frac{n\pi}{L} x$	$L \quad n = 0$ $\frac{L}{2} \quad n = 1, 2, \ldots$	$\frac{1}{\sqrt{L}}$ $\sqrt{\frac{2}{L}} \cos \frac{n\pi}{L} x$
Dirichlet $X(0) = 0$ Robin $k_2 X'(L) + h_2 X(L) = 0$	λ_n are positive roots of $\lambda \cos \lambda L + H_2 \sin \lambda L = 0$ $\quad n = 1, 2, \ldots$	$\sin \lambda_n x$	$\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\sin \lambda_n x}{\sqrt{\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Neumann $X'(0) = 0$ Robin $k_2 X'(L) + h_2 X(L) = 0$	λ_n are positive roots of $\lambda \sin \lambda L - H_2 \cos \lambda L = 0$ $\quad n = 1, 2, \ldots$	$\cos \lambda_n x$	$\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\cos \lambda_n x}{\sqrt{\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1 X'(0) + h_1 X(0) = 0$ Dirichlet $X(L) = 0$	λ_n are positive roots of $\lambda \cos \lambda L + H_1 \sin \lambda L = 0$ $\quad n = 1, 2, \ldots$	$\sin \lambda_n (x - L)$	$\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\sin \lambda_n (x - L)}{\sqrt{\frac{L}{2} - \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1 X'(0) + h_1 X(0) = 0$ Neumann $X'(L) = 0$	λ_n are positive roots of $\lambda \sin \lambda L - H_1 \cos \lambda L = 0$ $\quad n = 1, 2, \ldots$	$\cos \lambda_n (x - L)$	$\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}$	$\frac{\cos \lambda_n (x - L)}{\sqrt{\frac{L}{2} + \frac{\sin(2\lambda_n L)}{4\lambda_n}}}$
Robin $-k_1 X'(0) + h_1 X(0) = 0$ Robin $k_2 X'(L) + h_2 X(L) = 0$	λ_n are positive roots of $(H_1 H_2 - \lambda^2) \sin \lambda L + (H_1 + H_2) \lambda \cos \lambda L = 0$ $n = 1, 2, \ldots$	$\lambda_n \cos \lambda_n x + H_1 \sin \lambda_n x$	$\frac{(\lambda_n^2 + H_1^2)}{2}\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) + \frac{H_1}{2}$	$\frac{\lambda_n \cos \lambda_n x + H_1 \sin \lambda_n x}{\sqrt{\frac{(\lambda_n^2 + H_1^2)}{2}\left(L + \frac{H_2}{\lambda_n^2 + H_2^2}\right) + \frac{H_1}{2}}}$

$\{X_n(x)\}$ is a complete set of orthogonal functions on [0,L]

$$\int_0^L X_n(x) X_m(x)\,dx = \begin{cases} \lVert X_n \rVert^2 & n = m \\ 0 & n \neq m \end{cases}$$

finite Fourier transform

$$F_n = \int_0^L K_n(x) f(x)\,dx$$

generalized Fourier series

$$f(x) = \sum_n a_n X_n(x) \qquad a_n = \frac{\int_0^L X_n(x) f(x)\,dx}{\lVert X_n \rVert^2}$$

inverse transform (Fourier series)

$$f(x) = \sum_n F_n K_n(x)$$

laplace transform

Laplace transform $\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$f(t)$ is of exponential order if $|f(t)| \le Me^{at}$ $\forall t \ge 0$ for some $a, M > 0$

existance of Laplace transform: if $f(t)$ is picewise continuos on $[0, \infty)$ and of exponential order with a and M then $\mathcal{L}\{f(t)\}$ exists for all $s > a$

$$|\mathcal{L}\{f(t)\}| \le \frac{M}{s-a}$$

$\mathcal{L}\{f(t)\} \to 0$ when $s \to \infty$

$s\mathcal{L}\{f(t)\}$ is bounded when $s \to \infty$

PROPERTIES

linearity

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

integral

$$\mathcal{L}\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}F(s)$$

s-multiplied transform

$$\mathcal{L}^{-1}\{sF(s)\} = f'(t)$$

s-divided transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(x)dx$$

transform differentiation

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

$$\mathcal{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

transform integration

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} F(s)ds$$

$$\mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^{\infty} F(s)ds\right\}$$

similarity

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad a > 0$$

shifting on s

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{-at}\mathcal{L}^{-1}\{F(s-a)\}$$

$$\mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$$

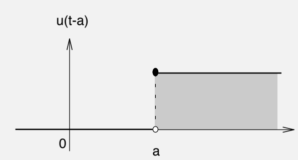
$$\mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$$

$$\mathcal{L}\{e^{-at}t^n\} = \begin{cases} \dfrac{\Gamma(n+1)}{(s+a)^{n+1}} & n > -1 \\ \dfrac{n!}{(s+a)^{n+1}} & n = 0,1,\ldots \end{cases}$$

shifting on t

unit step function

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \ge a \end{cases}$$



$$\mathcal{L}\{u(t-a)f(t)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \qquad a \ge 0$$

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

convolution

$$f * g = \int_0^t f(t-x)g(x)dx$$

transform of convolution

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

table of laplace transforms

$f(t)$ $t \ge 0$	$\Phi(s)$	$f(t)$ $t \ge 0$	$\Phi(s)$
1	$\frac{1}{s}$ $s > 0$	e^{at}	$\frac{1}{s-a}$ $s > a$
t	$\frac{1}{s^2}$ $s > 0$	te^{at}	$\frac{1}{(s-a)^2}$ $s > a$
t^n $n = 1,2,\ldots$	$\frac{n!}{s^{n+1}}$ $s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ $s > a$
t^a $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$ $s > 0$	$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
$\sin at$	$\frac{a}{s^2+a^2}$ $s > 0$	$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \ge a \end{cases}$ $a > 0$	$\frac{e^{-as}}{s}$ $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$ $s > 0$	$\delta(t)$	1
		$\delta(t-a)$ $a \ge 0$	e^{-as} $s > 0$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$ $s > 0$	$J_0(at)$	$\frac{1}{\sqrt{s^2+a^2}}$ $s > 0$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $s > 0$	$J_0(a\sqrt{t})$	$\frac{e^{-\frac{a^2}{4s}}}{s}$ $s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$ $s > a$	$J_n(at)$ $n = 0,1,2,..$	$\frac{\left(\sqrt{s^2+a^2}-s\right)^n}{a^n\sqrt{s^2+a^2}}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$ $s > a$	$t^p J_p(at)$ $p > -\frac{1}{2}$	$\frac{2^p a^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}(s^2+a^2)^{p+\frac{1}{2}}}$ $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$ $s > \lvert a \rvert$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(at)$ $k > 0$	$\frac{1}{(s^2+a^2)^k}$ $s > 0$
$\cosh at$	$\frac{s}{s^2-a^2}$ $s > \lvert a \rvert$	$\frac{\sqrt{\pi}}{\Gamma(k)} a \left(\frac{t}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{3}{2}}(at)$ $k > \frac{1}{2}$	$\frac{s}{(s^2+a^2)^k}$ $s > 0$
$t \sinh at$	$\frac{2bs}{(s^2-a^2)^2}$ $s > \lvert a \rvert$	$\mathrm{erf}(at)$ $a > 0$	$\frac{1}{s} e^{\frac{s^2}{4a^2}} \mathrm{erfc}\left(\frac{s}{2a}\right)$ $s > 0$
$t \cosh at$	$\frac{s^2+b^2}{(s^2-a^2)^2}$ $s > \lvert a \rvert$	$\mathrm{erf}(a\sqrt{t})$ $a \ge 0$	$\frac{a}{s\sqrt{s+a^2}}$ $s > 0$
$e^{at} \sinh bt$	$\frac{b}{(s-a)^2-b^2}$ $s > a + \lvert b \rvert$	$\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)$ $a \ge 0$	$\frac{1}{s} e^{-a\sqrt{s}}$ $s > 0$
$e^{at} \cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$ $s > a + \lvert b \rvert$	$e^{-a^2t^2}$ $a > 0$	$\frac{\sqrt{\pi}}{2a} e^{\frac{s^2}{4a^2}} \mathrm{erfc}\left(\frac{s}{2a}\right)$ $s > 0$

maple

For calcualtaion of Laplace transform or inverse Laplace transform the package with integral transforms has to be downloaded:

```
> with(inttrans);
   [fourier,laplace,invlaplace,...]
```

Laplace transform is calculated with the command laplace (f(t),t,s):

f(t) denotes the function to be transformed, t is the independent variable of the function, s is the variable of the transformed function

Example 1:

```
> laplace(t^2,t,s);
```

$$\frac{2}{s^3}$$

Example 2:

```
> f(t):=t^2*sin(5*t);
```

$$f(t) := t^2 \sin(5t)$$

```
> laplace(f(t),t,s);
```

$$\frac{10(3s^2-25)}{(s^2+25)^3}$$

Inverse Laplace transform is calculated with the command invlaplace (Φ(s),s,t):

Φ(s) denotes the function to be transformed, s is the independent variable of the function, t is the variable of the transformed function

Example 3:

```
> phi(s):=exp(-4*s)/s;
```

$$\Phi(s) := \frac{e^{(-4s)}}{s}$$

```
> laplace( Φ(s),s,t);
```

$$\mathrm{Heaviside}(t-4)$$

Example 4:

```
> phi(s):=exp(-3*sqrt(s));
```

$$\Phi(s) := e^{(-3\sqrt{s})}$$

```
> laplace( Φ(s),s,t);
```

$$\frac{3e^{\left(-\frac{9}{4t}\right)}}{2\sqrt{\pi}\,t^{(3/2)}}$$

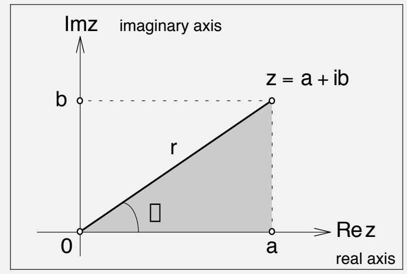
complex numbers

complex number	$z \in C$	we need complex numbers to be able to solve algebraic equations such as $x^2+1=0$ which has no solution in real numbers
standard form	$z = a + ib$	where i is imaginary unit with property $i^2=-1$ and a and b are real numbers $a, b \in R$
	$\operatorname{Re} z = a$	real part of z
	$\operatorname{Im} z = b$	imaginary part of z
exponential (polar) form	$z = a + ib = re^{i\varphi}$	
trigonometric form	$z = a + ib = r(\cos\varphi + i\sin\varphi)$	
vector form	$z = (a,b)$	
Euler's formula	$e^{i\varphi} = \cos\varphi + i\sin\varphi$	

complex plane

any point in a plane with coordinates (a,b) is associated with a complex number a+ib

polar coordinates (r,φ) of the point (a,b) can be used for representation of complex numbers



absolute value or modulus of z	$r = \lvert z \rvert$	conversion formulas:	$r^2 = a^2 + b^2$	$a = r\cos\varphi$
amplitude or argument of z	$\varphi = \arg z$		$\tan\varphi = \dfrac{b}{a}$	$b = r\sin\varphi$

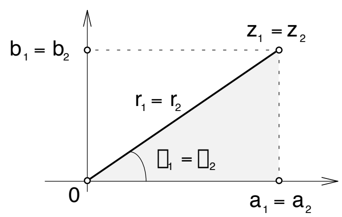
trigonometric functions in complex form

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

algebra of complex numbers

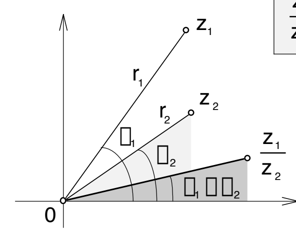
$$z_1 = a_1 + ib_1 = r_1 e^{i\varphi_1} \qquad z_2 = a_2 + ib_2 = r_2 e^{i\varphi_2}$$

equality



$$z_1 = z_2 \iff \begin{matrix} b_1 = b_2 \\ a_1 = a_2 \end{matrix} \iff \begin{matrix} r_1 = r_2 \\ \varphi_1 = \varphi_2 \end{matrix}$$

quotent

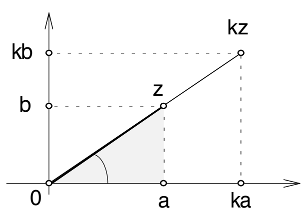


$$\frac{z_1}{z_2} = \frac{(a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$= \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right]$$

multiplication by a scalar



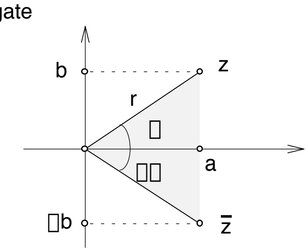
$$kz = ka + i(kb)$$

$$= kre^{i\varphi}$$

$$= kr(\cos\varphi + i\sin\varphi)$$

$$= (ka, kb)$$

conjugate



$$\bar{z} = a - ib$$

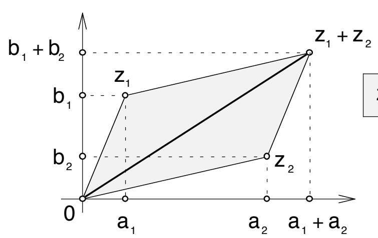
$$= re^{-i\varphi}$$

$$= r(\cos\varphi - i\sin\varphi)$$

$$= (a, -b)$$

$$z\bar{z} = a^2 + b^2 \qquad r = \lvert z \rvert = \sqrt{z\bar{z}}$$

sum

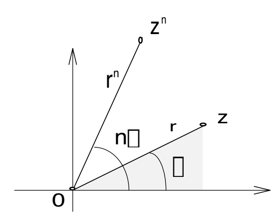


$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$= r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2}$$

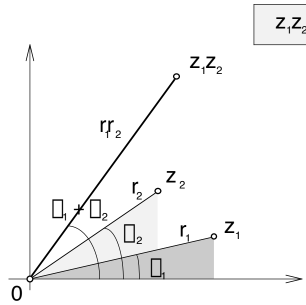
$$= (a_1 + a_2, b_1 + b_2)$$

powers (De Moivre's Formula)



$$z^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i\sin n\varphi)$$

product



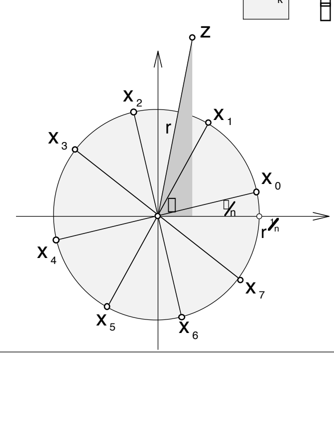
$$z_1z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$$

$$= r_1r_2 e^{i(\varphi_1 + \varphi_2)}$$

$$= r_1r_2\left[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right]$$

$$= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$$

roots



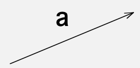
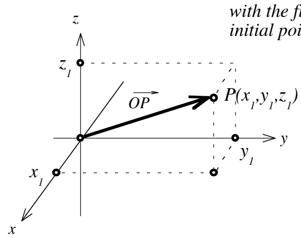
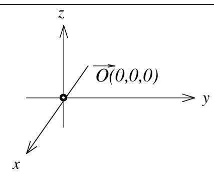
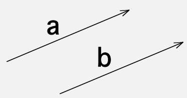
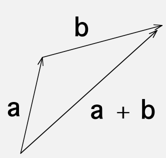
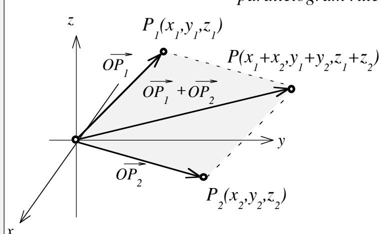
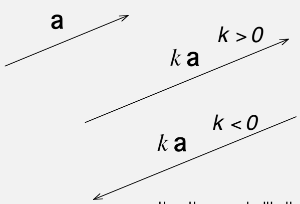
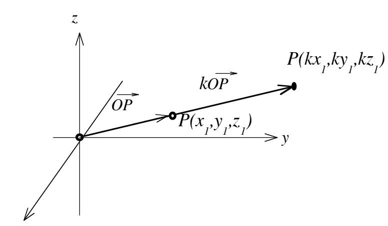
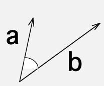
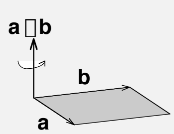
$$x_k = \left(z^{\frac{1}{n}}\right)_k = r^{\frac{1}{n}} e^{i\frac{\varphi + k2\pi}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{\varphi}{n} + \frac{k2\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{k2\pi}{n}\right)\right]$$

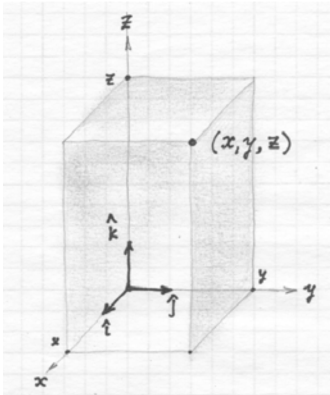
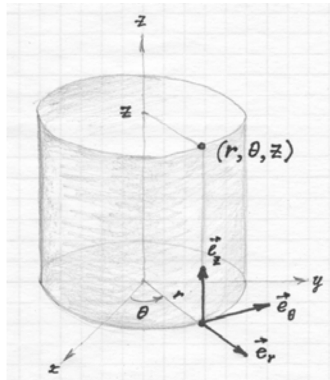
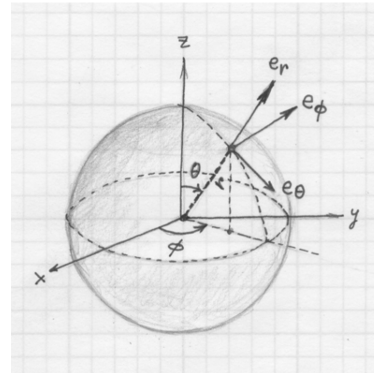
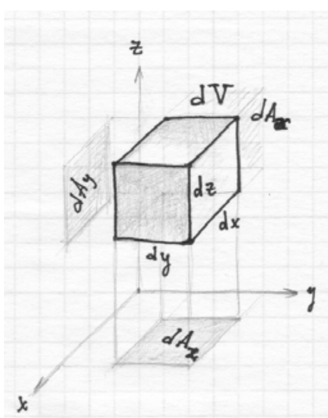
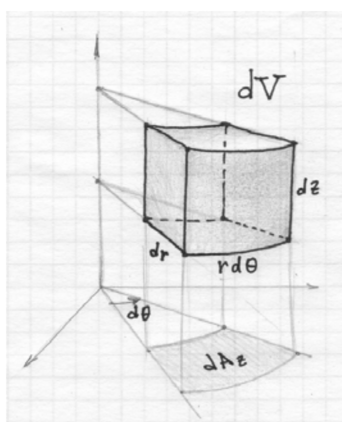
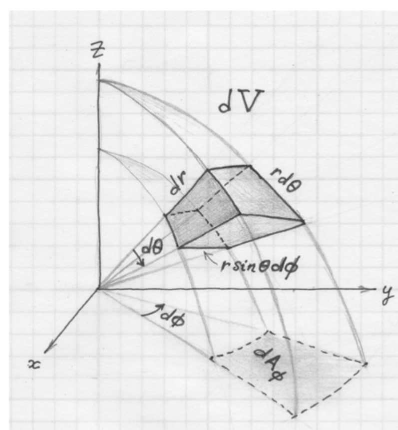
$$k = 0, 1, 2, \ldots, n-1$$

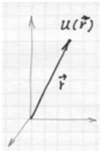
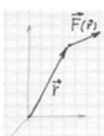
- $z^{1/n}$ can be treated as the solutions of algebraic equation $x^n=z$ which has exactly n roots
- all roots are evenly distributed on the circle with radius $r^{1/n}$
- if z is a real number, then complex roots appear in conjugate pairs

Vectors in Euclidian Space

free vector	position vector	coordinate vector	1st order tensor
 *directed segment*	*directed segment with the fixed initial point* 	*triple of real numbers* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$	a_i with index convention: $i = 1, 2, 3$ and summation convention: $a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$ $\delta_{ij} a_j = a_1 + a_2 + a_3$
zero vector ∘ *any point*		$\mathbf{0} = \langle 0,0,0 \rangle$	0
norm $\lVert \mathbf{a} \rVert$ = *length of segment*	$\lVert \vec{OP} \rVert = \sqrt{x_1^2 + y_1^2 + z_1^2}$	$\lVert \mathbf{a} \rVert = \sqrt{a_1^2 + a_2^2 + a_3^2}$	$\sqrt{\delta_{ij} x_i x_j}$
equality $\mathbf{a} = \mathbf{b}$  *two vectors are equal if they have the same norm and direction*	$\vec{OP_1} = \vec{OP_2} \Leftrightarrow$ $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$	$\mathbf{a} = \mathbf{b} \Leftrightarrow$ $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$	$a_i = b_i$
summation 	*parallelogram rule* 	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$	$a_i + b_i$
multiplication by a scalar  $\lVert k\mathbf{a} \rVert = \lvert k \rvert \lVert \mathbf{a} \rVert$		$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$	ka_i
dot product  $\mathbf{a} \cdot \mathbf{b} = \lVert \mathbf{a} \rVert \lVert \mathbf{b} \rVert \cos(\mathbf{a}, \mathbf{b})$	$\vec{OP_1} \cdot \vec{OP_2} = x_1 x_2 + y_1 y_2 + z_1 z_2$	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$	$a_i b_i$
cross product 	$\vec{OP_1} \times \vec{OP_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$	$\mathbf{a} \times \mathbf{b}$ $= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$	$(\mathbf{a} \times \mathbf{b})_i = a_j b_k - a_k b_j$ i,j,k is cyclic permutation of 1,2,3

Coordinate Systems

Cartesian coordinates (x, y, z)	Cylindrical coordinates (r, θ, z)	Spherical coordinates (r, ϕ, θ)
	 $x = r\cos\theta$ $y = r\sin\theta$ $z = z$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$ $z = z$	 $x = r\cos\phi\sin\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\theta$ $r^2 = x^2 + y^2 + z^2$ $\tan\phi = \frac{y}{x}$ $\tan\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$
Basic vectors $\mathbf{i} = (1,0,0)$ $\mathbf{j} = (0,1,0)$ $\mathbf{k} = (0,0,1)$	$\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ $\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$ $\mathbf{e}_z = \mathbf{k}$	$\mathbf{e}_r = \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta$ $\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\phi$ $\mathbf{e}_\phi = \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta$
 Line elements dx, dy, dz Differential areas $dA_x = dydz$ $dA_y = dxdz$ $dA_z = dxdy$ Differential volume $dV = dxdydz$ Arc length $ds^2 = dx^2 + dy^2 + dz^2$	 $dr, rd\theta, dz$ $dA_r = rd\theta dz$ $dA_\theta = drdz$ $dA_r = rd\theta dr$ $dV = rdrd\theta dz$ $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$	 $dr,\ r\sin\theta d\phi,\ rd\theta$ $dA_r = r^2\sin\theta d\phi d\theta$ $dA_\phi = r\sin\theta d\phi dr$ $dA_\phi = \rho d\phi d\rho$ $dV = r^2\sin\theta d\phi d\theta dr$ $ds^2 = dr^2 + r^2\sin^2\theta d\phi + r^2 d\theta^2$

scalar field $u(\mathbf{r})$ 	$u(x,y,z)$	$u(r,\theta,z)$	$u(r,\phi,\theta)$
Gradient ∇u	$\nabla u = \left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$ $= \frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}$	$\nabla u = \left(\frac{\partial u}{\partial r},\frac{1}{r}\frac{\partial u}{\partial \theta},\frac{\partial u}{\partial z}\right)$ $= \frac{\partial u}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta+\frac{\partial u}{\partial z}\mathbf{e}_z$	$\nabla u = \left(\frac{\partial u}{\partial r},\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi},\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$ $= \frac{\partial u}{\partial r}\mathbf{e}_r+\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\mathbf{e}_\phi+\frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta$
Laplacian $\nabla^2 u$	$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$	$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+$ $+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$	$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+$ $+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)$
vector field $\mathbf{F}(\mathbf{r})$ 	(F_x,F_y,F_z)	(F_r,F_θ,F_z) $F_r = F_x\cos\theta+F_y\sin\theta$ $F_\theta = F_x\sin\theta+F_y\cos\theta$ $F_z = F_z$ $F_x = F_r\cos\theta-F_\theta\sin\theta$ $F_y = F_r\sin\theta+F_\theta\cos\theta$ $F_z = F_z$	(F_r,F_ϕ,F_θ) $F_r = F_x\cos\phi\sin\theta+F_y\sin\phi\sin\theta+F_z\cos\theta$ $F_\phi = -F_x\sin\phi+F_y\cos\phi$ $F_\theta = F_x\cos\phi\cos\theta+F_y\sin\phi\cos\theta-F_z\sin\theta$ $F_x = F_r\cos\phi\sin\theta-F_\phi\sin\phi+F_\theta\cos\phi\cos\theta$ $F_y = F_r\sin\phi\sin\theta+F_\phi\cos\phi+F_\theta\sin\phi\cos\theta$ $F_z = F_r\cos\theta-F_\theta\sin\theta$
Divergence $div\mathbf{F} = \nabla\cdot\mathbf{F}$	$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{1}{r}\frac{\partial F_\theta}{\partial \theta}+\frac{\partial F_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}+$ $+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left[(\sin\theta)F_\theta\right]$
$curl\mathbf{F} = \nabla\times\mathbf{F}$	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix} =$ $\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j}+$ $+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}$	$\frac{1}{r}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ F_r & rF_\theta & F_z\end{vmatrix} =$ $\left(\frac{1}{r}\frac{\partial F_z}{\partial \theta}-\frac{\partial F_\theta}{\partial z}\right)\mathbf{e}_r+$ $+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\mathbf{e}_\theta+$ $\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial \theta}\right]\mathbf{e}_z$	$\frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix} =$ $\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial \theta}-\frac{\partial F_\theta}{\partial \phi}\right]\mathbf{e}_r+$ $+\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial \theta}\right]\mathbf{e}_\phi+$ $+\frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial \phi}-\sin\theta\frac{\partial(rF_\phi)}{\partial r}\right]\mathbf{e}_\theta$

bessel functions

bessel equation (BE)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

BE can be obtained from eqn $r^2R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0$ by the change of variable $y(x) = R(r)$, $x = \lambda r$

solutions of BE are

bessel function of the 1st kind of order ν

when $\nu \neq$ *integer*

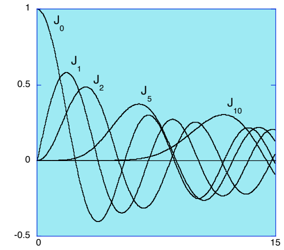
$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}$$

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)}$$

when $n = 1,2,\ldots$ functions of integer order

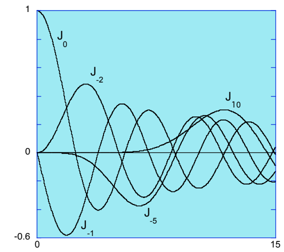
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!\,(n+k)!}$$

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k-n}}{k!\,(-n+k)!}$$



when $\nu \neq$ *integer* $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent

general solution of BE $\quad y(x) = c_1J_\nu(x) + c_2J_{-\nu}(x)$

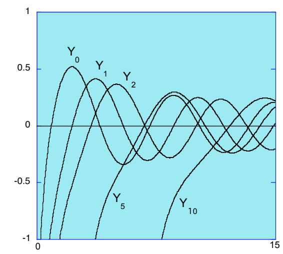


bessel function of the 2nd kind of order ν

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad Y_n(x) = \lim_{\nu\to n} Y_\nu(x) \qquad Y_{-n}(x) = (-1)^n Y_n(x)$$

$J_\nu(x)$ and $Y_\nu(x)$ are always linearly independent (when ν is integer or not)

general solution of BE $\quad y(x) = c_1J_\nu(x) + c_2Y_\nu(x)$



properties

(the same properties also hold for bessel functions of the 2nd kind Y)

$$J_{-n}(x) = (-1)^n J_n(x) \qquad J_{-n}(x) = J_n(-x)$$

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x) \qquad J_{\nu-1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x}J_\nu(x) \qquad J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x}J_\nu(x)$$

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x) \qquad \frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$$

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) \qquad \int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_\nu(x)$$

orthogonality

functions $y_\nu(\lambda_n x)$ are combinations of $J_\nu(\lambda_n x)$ and $Y_\nu(\lambda_n x)$

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the values of parameter λ (eigenvalues) for which boundary-value problem (Sturm-Liouville problem)

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \qquad x \in (x_1, x_2)$$

$$a_1y(x_1) - b_1y'(x_1) = 0 \qquad a_1^2 + b_1^2 \neq 0$$

$$a_2y(x_2) - b_2y'(x_2) = 0 \qquad a_2^2 + b_2^2 \neq 0$$

has non-trivial solutions (eigenfunctions)

$y_\nu(\lambda_1x), y_\nu(\lambda_2x), y_\nu(\lambda_3x), \ldots$

then $\{y_\nu(\lambda_nx)\}$ $n = 1,2,\ldots$ is a complete set of functions orthogonal on (x_1, x_2) w.r.t. weight x

$$\int_{x_1}^{x_2} x\,y_\nu(\lambda_nx)\,y_\nu(\lambda_mx)\,dx = 0 \quad \text{when } n \neq m$$

bessel-fourier series

$$f(x) = \sum_{n=1}^{\infty} c_n y_\nu(\lambda_nx)$$

$$c_n = \frac{\int_{x_1}^{x_2} x\,y_\nu(\lambda_nx)f(x)\,dx}{\int_{x_1}^{x_2} x\,y_\nu^2(\lambda_nx)\,dx}$$

modified bessel equation (MBE)

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

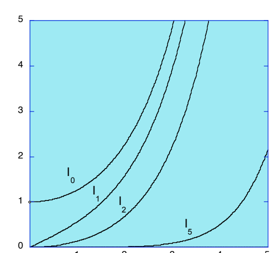
solutions of MBE are

modified bessel function of the 1st kind of order ν

$$I_\nu(x) = (-1)^{\nu}J_\nu(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)} \qquad I_{-\nu}(x) = (-1)^{-\nu}J_{-\nu}(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k-\nu}}{k!\,\Gamma(-\nu+k+1)}$$

when $\nu \neq$ *integer* $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly independent

general solution of MBE $\quad y(x) = c_1I_\nu(x) + c_2I_{-\nu}(x)$

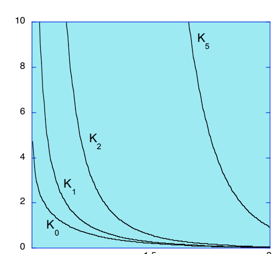


modified bessel function of the 2nd kind of order ν

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi} \qquad K_n(x) = \lim_{\nu\to n} K_\nu(x)$$

$I_\nu(x)$ and $K_\nu(x)$ are always linearly independent

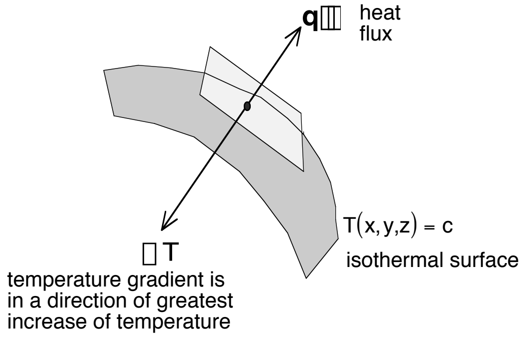
general solution of MBE $\quad y(x) = c_1I_n(x) + c_2K_n(x)$



Fourier's law

heat conduction in continuous medium

$$\mathbf{q}'' = -k\nabla T$$

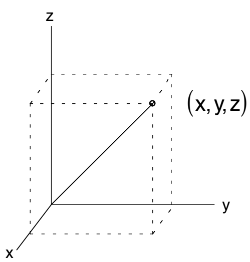


k	$\left[\frac{W}{m\cdot K}\right]$	coefficient of thermal conductivity
h	$\left[\frac{W}{m^2\cdot K}\right]$	coefficient of convective heat transfer
$\alpha = \frac{k}{\rho c_p}$	$\left[\frac{m^2}{s}\right]$	thermal diffusivity
$\dot{q}$	$\left[\frac{W}{m^3}\right]$	heat generation per unit volume
$q_g = \dot{q}V$	$[W]$	rate of heat generation
ε		surface emissivity

Heat Equation

Cartesian coordinates

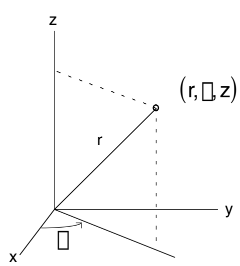
$$\mathbf{q}'' = \left\langle -k\frac{\partial T}{\partial x}, -k\frac{\partial T}{\partial y}, -k\frac{\partial T}{\partial z}\right\rangle$$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

cylindrical coordinates

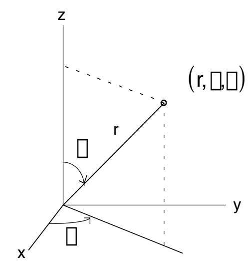
$$\mathbf{q}'' = \left\langle -k\frac{\partial T}{\partial r}, -\frac{k}{r}\frac{\partial T}{\partial \phi}, -k\frac{\partial T}{\partial z}\right\rangle$$



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

spherical coordinates

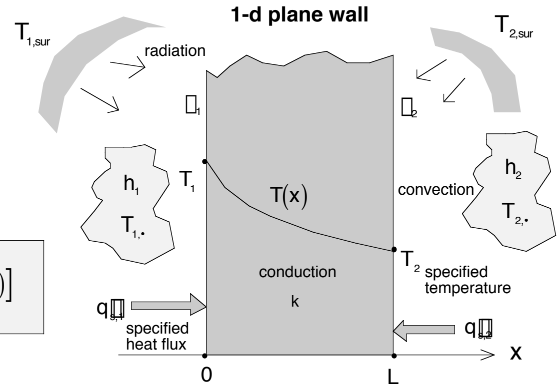
$$\mathbf{q}'' = \left\langle -k\frac{\partial T}{\partial r}, -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}, -\frac{k}{r}\frac{\partial T}{\partial \theta}\right\rangle$$



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary Conditions

1-d plane wall



non-linear boundary condtions:

at $x = 0$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h_1\left[T_{1,\infty} - T(0)\right] + \varepsilon_1\sigma\left[T_{1,sur}^4 - T^4(0)\right]$$

at $x = L$

$$k\frac{\partial T}{\partial x}\bigg|_{x=L} = h_2\left[T_{2,\infty} - T(L)\right] + \varepsilon_2\sigma\left[T_{2,sur}^4 - T^4(L)\right]$$

$\sigma = 5.67e-8 \left[\frac{W}{m^2K^4}\right]$ Stefan - Boltzmann constant

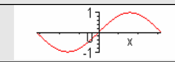
classification of linearized boundary condtions:

I Dirichlet — $T\big|_{x=0} = T_1$ — constant surface temperature — $T\big|_{x=L} = T_2$

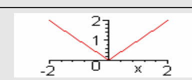
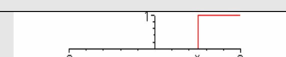
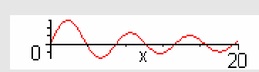
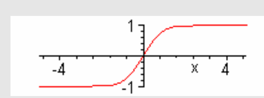
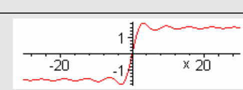
II Neumann — $-k\frac{\partial T}{\partial x}\big|_{x=0} = q''_{s,1}$ — constant heat flux at the wall — $k\frac{\partial T}{\partial x}\big|_{x=L} = q''_{s,2}$ — $\frac{\partial T}{\partial x}\big|_{x=0} = 0$ adiabatic surface: perfectly insulated surface (no flux thru the wall)

III Robin — $\left(-k\frac{\partial T}{\partial x} + h_1 T\right)\big|_{x=0} = f_1$ — convective boundary condition — $\left(k\frac{\partial T}{\partial x} + h_2 T\right)\big|_{x=L} = f_2$

SYMBOLS AND OPERATORS

SYMBOL	DESCRIPTION	EXAMPLE	RESULT
restart	clears all definitions	> `restart;`	
with	loads Maple packages: *linalg,plots,DEtools,PDEtools,student*	> `with(plots);` shows all commands in the package	*[animate,animae3d,...*
;	execute and show result	> `2+3;`	5
:	execute and hide result	> `5-2:`	
..	range or interval	> `plot(sin(x),x=-Pi..Pi);`	
( )	grouping in arithmetic expressions	> `(3+2)*5;`	25
[ ]	list delimiter	> `v:=vector([1,0,-2]);`	v:=[1,0,-2]
{ }	set delimiter	> `{f(x),g(x)};`	{f(x),g(x)};
:=	assigment	> `f(x):=cos(x);`	f(x):=cos(x);
=	equal	> `subs(x=Pi,cos(x));`	cos(π)
%	refers to previous result		
evalf	evaluate	> `evalf(cos(Pi));`	-1.
simplify	simplifies expressions	> `simplify(x^a*x^b,power);`	x^{a+b}
<, <= >, >= <>	less than, less than or equal greater than, greater than or equal not equal		
+	addition	> `2+4;`	6
-	subtraction	> `7-2;`	5
*	multiplication	> `2*3;`	6
/	division	> `10/2;`	5
^	exponentiation	> `x^2;`	x^2
->	defines function (mapping)	> `f:=x->sqrt(1-x)`	$f := x \rightarrow \sqrt{1-x}$
Pi,exp(1),infinity	mathematical constants	> `evalf(Pi);`	3.141592654
I	imaginary unit	> `sqrt(-1);`	*I*

FUNCTIONS

exp	natural exponential function	> `exp(x);`	e^x
ln log10, log[a]	natural logarithmic function logarithmic function base 10, base a	> `ln(2.0);` > `log10(2.0);`	.6931471806 .3010299957
sin,cos tan,cot sec,csc	trigonometric functions	> `sin(Pi/3);`	$\frac{\sqrt{3}}{2}$
arcsin, arcos arctan	inverse trigonometric functions	> `arcsin(1);`	$\frac{\pi}{2}$
sinh,cosh tanh,coth sech,csch	hyperbolic functions	> `cosh(2.5);`	6.132289480
sqrt	square root	> `sqrt(2.0);`	1.414213562
abs	absolute value function	> `plot(abs(x),x=-2..2);`	
Heaviside	Heaviside's function	> `plot(Heaviside(x-1,x=2..2);`	
BesselJ(n,x) BesselY(n,x) BesselI(n,x) BesselK(n,x)	Bessel functions of order n	> `plot(BesselJ(1,x),x=0..20);`	
Dirac	Dirac delta function	> `int(Dirac(x),x=-1..1);`	1
erf erfc	error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ complimentary $erfc(x) = 1 - erf(x)$	> `plot(erf(x),x=-5..5);`	
hypergeom	hypergeometric function		
factorial !	factorial of an integer	> `factorial(5);` > `6!;`	120 720
Si	sine integral $Si(x) = \int_0^x \frac{\sin t}{t} dt$	> `plot(Si(x),x=-30..30);`	

Prepared by Vladimir Solovjov, BYU, 2002 (modified 2004) vps@et.byu.edu

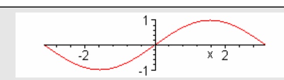
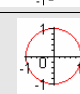
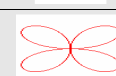
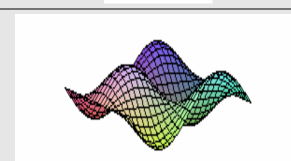
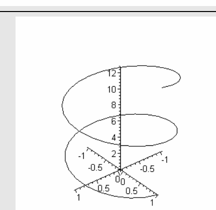
CALCULUS

SYMBOL	DESCRIPTION	EXAMPLE	RESULT
diff	derivative	`> diff(x*exp(x),x);`	$e^x + x\,e^x$
	n^{th} derivative	`> diff(x*exp(x),x$3);`	$3\,e^x + x\,e^x$
int	definite integral indefinite integral	`> int(sin(x),x=0..Pi/2);` `> int(ln(x),x);`	1 $x\ln(x) - x$
simplify	simplify expression		
subs	substitute	`> subs(t=0,u(x,t));`	$u(x,0)$
factor	factor a polynomial	`> factor(x^4-x^2);`	$x^2(x-1)(x+1)$
limit	limit	`> limit(sin(x)/x,x=0);`	1
Diff, Int, Limit, Sum value	inert form of operators evaluate an inert expression	`> Int(x^2,x);` `> value(%);`	$\int x^2\,dx$ $\frac{x^3}{3}$
convert	convert expression in		
	partial fractions	`> convert((x^2+1)/(x^3-x) parfrac,x);`	$-\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}$
	Euler formula	`> convert(exp(b*I),trig);`	$\cos(b)+I\sin(b)$
sum	summation	`> sum(u[n](x),n=1..4);`	$u_1(x) + u_2(x) + u_3(x) + u_4(x)$
series	Macloren series	`> series(exp(x),x,4);`	$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$
taylor	Taylor series	`> taylor(ln(x),x=1,3);`	$x - 1 - \frac{1}{2}(x-1)^2 + O((x-1)^3)$

SOLVERS

solve	general equation solver	`> solve({x^2-a=0},{x});`	$\{x = \sqrt{a}\}, \{x = -\sqrt{a}\}$
fsolve	numeric equation solver	`> fsolve(x*tan(x)=1,x=0..1);`	.860333589[illegible]
dsolve	solution of differential equation	`> s:=dsolve({diff(y(x),x)= x+1,y(0)=0});`	$s := y(x) = \frac{1}{2}x^2 + x$
unapply	produces the function from expression	`> f:=unapply(x^2/2+x,x);`	$f := x \to \frac{1}{2}x^2 + x$
assign	turns sign "=" in the solution set into ":=" does not create functions	`> assign(s);` `> y(x);`	$\frac{1}{2}x^2 + x$

PLOTS

plot	plot of function y=f(x)	`> plot(sin(x),x=-Pi..Pi);`	
	parametric plot plot([x(t),y(t),t=a..b])	`> plot([sin(x),cos(x),x=0..2*Pi]);`	
polarplot	polarplot($r(\theta), \theta = \alpha..\beta$)	`> polarplot(sin(2*x),x=0..2*Pi);`	
plot3d	plot3d(f(x,y),x=a..b,y=c..d)	`> plot3d(sin(x)*cos(y), x=0..2*Pi,y=0..2*Pi);`	
spacecurve	parametric curve	`> with(plots):` `> spacecurve([sin(t),cos(t),t], t=0..4*Pi,axes=normal,color=black);`	

Greek Letters

α	Α	alpha	ν	Ν	nu
β	Β	beta	ξ	Ξ	ksi
γ	Γ	gamma	ο	Ο	omikron
δ	Δ	delta	π	Π	pi
ε	Ε	epsilon	ρ	Ρ	rho
ς	Ζ	zeta	σ	Σ	sigma
η	Η	eta	τ	Τ	tau
ϑ	Θ	theta	υ	ϒ	upsilon
ι	Ι	iota	φ	Φ	phi
κ	Κ	kappa	χ	Χ	chi
λ	Λ	lambda	ψ	Ψ	psi
μ	Μ	mu	ω	Ω	omega

How to Write Greek Letters

The arrows show you where to start when you write Greek letters.