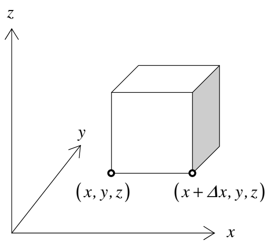
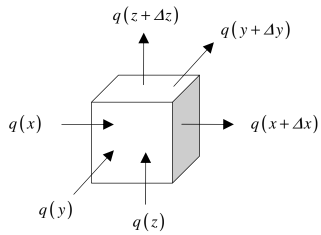
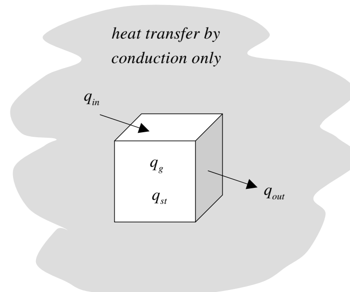
2.3 **THE HEAT EQUATION**

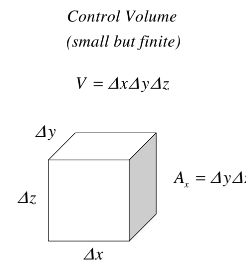
Derivation of the Heat Equation in the Cartesean coordinates (finite control volume approach)

Consider a Control Volume
in the stagnant continuous medium









Energy Balance:

$$q_{in} - q_{out} + q_g = q_{st}$$

$$q(x) - q(x+\Delta x) \ + \ q(y) - q(y+\Delta y) \ + \ q(z) - q(z+\Delta z) \ + \ q_g \ = \ q_{st}$$

$q_x = q''_x \cdot A_x = q''_x \cdot \Delta y \cdot \Delta z$
$q_y = q''_y \cdot A_y = q''_y \cdot \Delta x \cdot \Delta z$
$q_z = q''_z \cdot A_z = q''_z \cdot \Delta x \cdot \Delta y$

$$\left[q''(x) - q''(x+\Delta x)\right]\Delta y\Delta z \ + \ \left[q''(y) - q''(y+\Delta y)\right]\Delta x\Delta z \ + \ \left[q''(z) - q''(z+\Delta z)\right]\Delta x\Delta y \ + \ \dot{q}\Delta x\Delta y\Delta z \ = \ \rho c_p \frac{\partial T}{\partial t}\Delta x\Delta y\Delta z$$

divide equation
by ΔxΔyΔz

$$-\frac{q''(x+\Delta x) - q''(x)}{\Delta x} - \frac{q''(y+\Delta y) - q''(y)}{\Delta y} - \frac{q''(z+\Delta z) - q''(z)}{\Delta z} + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

take a limit when
$\Delta x \to 0$
$\Delta y \to 0$
$\Delta z \to 0$

$$-\frac{\partial q''_x}{\partial x} - \frac{\partial q''_y}{\partial y} - \frac{\partial q''_z}{\partial z} + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

apply Fourier's Law:
$q''_x = -k\dfrac{\partial T}{\partial x}$
$q''_y = -k\dfrac{\partial T}{\partial y}$
$q''_z = -k\dfrac{\partial T}{\partial z}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

In general, k is a tensor with components:
$k_x(T)$
$k_y(T)$
$k_z(T)$
depending on temperature

assume :
k = const

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{\rho c_p}{k}\frac{\partial T}{\partial t}$$

introduce :
$\alpha = \dfrac{k}{\rho c_p}$

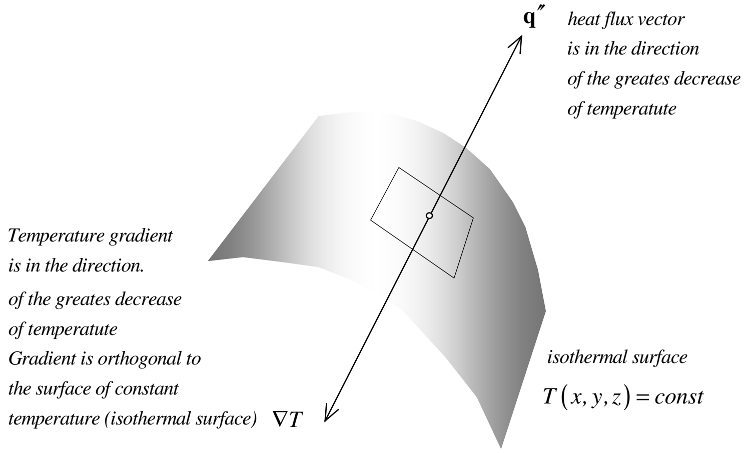
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The Heat Equation

$T(x), \ \dot{q}(x)$

2.1 FOURIER'S LAW

$$\mathbf{q}'' = -k\nabla T$$

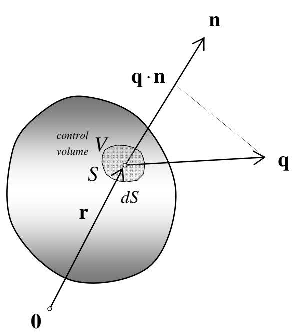


The Heat Equation - Derivation

Consider a point in the system defined by a position vector $\mathbf{r} \in D \subset \mathbb{R}^3$.
Let V be an arbitrary small control volume containing the point $\mathbf{r}$.

Conservation of energy principle for control volume V:

$$\begin{bmatrix} \text{rate of} \\ \text{net heat flow} \\ \text{through the} \\ \text{boundary } S \end{bmatrix} + \begin{bmatrix} \text{rate of heat} \\ \text{generation} \\ \text{in volume } V \end{bmatrix} = \begin{bmatrix} \text{rate of heat} \\ \text{storage} \\ \text{in volume } V \end{bmatrix}$$



$$-\int_S \mathbf{q}(\mathbf{r},t)\cdot\mathbf{n}\,dS \quad + \int_V g(\mathbf{r},t)\,dV \quad = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t}dV$$

Apply the divergence theorem and combine the terms:

$$\int_V \left[-\nabla\cdot\mathbf{q}(\mathbf{r},t) + g(\mathbf{r},t) - \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} \right] dV = 0$$

Because V is an arbitrary control volume, in the limit when $V \to 0$:

$$-\nabla\cdot\mathbf{q}(\mathbf{r},t) + g(\mathbf{r},t) - \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = 0$$

Apply the Fourier Law:

$$\nabla\cdot\left[k(\mathbf{r})\nabla T(\mathbf{r},t)\right] + g(\mathbf{r},t) = \rho(\mathbf{r})c_p(\mathbf{r})\frac{\partial T(\mathbf{r},t)}{\partial t}$$

If k, ρ, c_p are constant, then

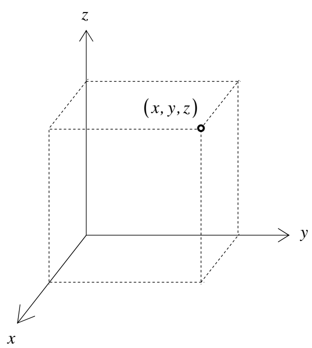
$$\nabla^2 T(\mathbf{r},t) + \frac{g(\mathbf{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t}$$

2.3 **THE HEAT DIFFUSION EQUATION** **in uniform medium:** k, ρ, c_p, and α are constants

Vector form

$$\nabla^2 T(\mathbf{r},t) + \frac{g(\mathbf{r},t)}{k} = \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t}$$

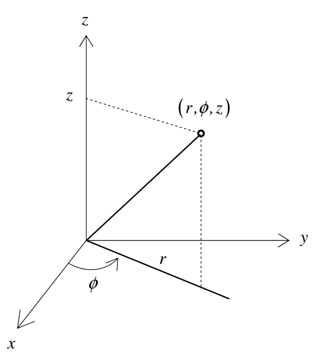
Cartesian coordinates



$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.21)$$

Cylindrical coordinates

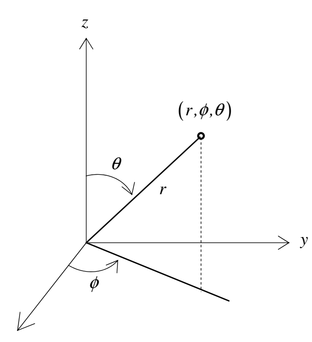


$$\nabla T = \left\langle \frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial T}{\partial \phi}, \frac{\partial T}{\partial z} \right\rangle$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.26)$$

$$r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Spherical coordinates



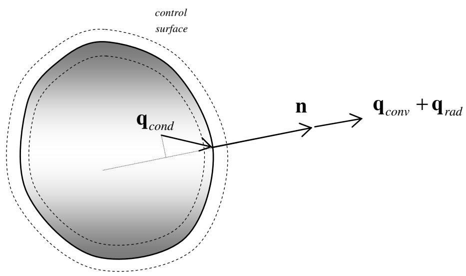
$$\nabla T = \left\langle \frac{\partial T}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}, \frac{1}{r}\frac{\partial T}{\partial z} \right\rangle$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta} \right) + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.29)$$

$$\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta} \right) + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Derivation of the Boundary Condition

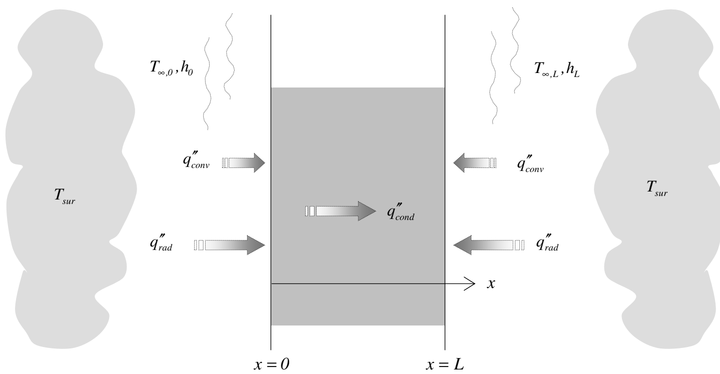
Energy balance for Control Surface:



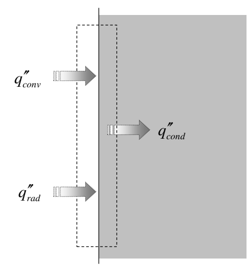
$$\mathbf{q}_{cond}\left(\mathbf{r},t\right)\cdot\mathbf{n}\Big|_S = h\Big[T\left(\mathbf{r},t\right)\Big|_S - T_\infty\Big] + \varepsilon\sigma\Big[T^4\left(\mathbf{r},t\right)\Big|_S - T_{sur}^4\Big]$$

2.4 BOUNDARY CONDITIONS in 1-D Cartesian Coordinates

Consider the boundaries x = 0 and x = L exposed to convective and radiative enviroment



Energy Balance for Control Surface:



Energy Balance:

$$q''_{cond} = q''_{conv} + q''_{rad}$$

$$\left[-k\frac{dT}{dx}\right]_{x=0} = h\Big[T_\infty - T\left(0\right)\Big] + \varepsilon\sigma\Big[T_{sur}^4 - T^4\left(0\right)\Big]$$

non-linear boundary condition

$$\left[-k\frac{dT}{dx} + hT\right]_{x=0} = hT_\infty$$

linear convective boundary condition with $h = h_{conv} + h_{rad}$ effective convective coefficient when it is assumed that $T_{sur} = T_\infty$

Linearization of the boundary conditions:

$$\varepsilon\sigma\Big[T_{sur}^4 - T^4\left(0\right)\Big] = \varepsilon\sigma\Big[T_{sur}^2 + T^2\left(0\right)\Big]\Big[T_{sur} + T\left(0\right)\Big]\Big[T_{sur} - T\left(0\right)\Big]$$
$$= h_{rad}\Big[T_{sur} - T\left(0\right)\Big]$$

$$h_{rad} = \varepsilon\sigma\Big[T_{sur}^2 + T^2\left(0\right)\Big]\Big[T_{sur} + T\left(0\right)\Big]$$

if $\dfrac{\left|T - T_{sur}\right|}{T_{sur}} \ll 1$ *then* $h_{rad} \approx 4\varepsilon\sigma T_{sur}^3$

Dirichlet	**I**	$\left[T\right]_{x=0} = T_{s,0}$	$\left[T\right]_{x=L} = T_{s,L}$	*prescribed temperature at the boundary* *(thermostated boundary with the surface temperature $T_s = T_\infty$ when $h \gg k$)*
Neumann	**II**	$\left[-k\dfrac{dT}{dx}\right]_{x=0} = q''_{s,0}$	$\left[k\dfrac{dT}{dx}\right]_{x=L} = q''_{s,L}$	*prescribed heat flux at the boundary* $\left[\dfrac{dT}{dx}\right]_{x=0} = 0$ *insulated boundary when $k \gg h$*
Robin	**III**	$\left[-k\dfrac{dT}{dx} + h_0 T\right]_{x=0} = h_0 T_{\infty,0}$	$\left[k\dfrac{dT}{dx} + h_L T\right]_{x=L} = h_L T_{\infty,L}$	*convective boundary condition*

INITIAL CONDITION

$$T\left(x,y,z,0\right) = T_0\left(x,y,z\right)$$

for all interior points of the domain $\left(x,y,z\right) \in D$