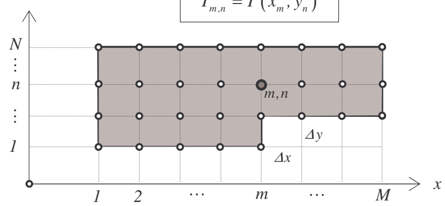
FINITE–DIFFERENCE METHOD — 2-D STEADY STATE CONDUCTION

Heat Equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$T_{m,n} = T(x_m, y_n)$ $n = 1,2,\dots,N \quad m = 1,2,\dots,M$

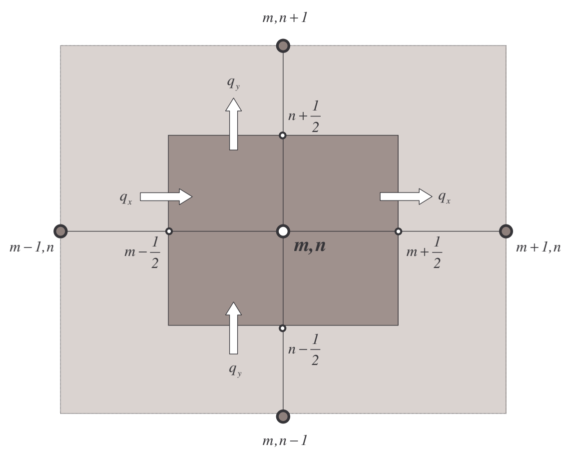


each point in a mesh is determined by a pair of indicies: m,n

The nodal network: temperature field $T(x,y)$ has to be determined only at the finite number of points (nodes)

$T_{m,n} \quad n = 1,2,\dots,N \quad m = 1,2,\dots,M$

Interior Nodes:



Central-difference approximation:

$$\left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2}} \approx \frac{T_m - T_{m-1}}{\Delta x} \qquad \left.\frac{\partial T}{\partial y}\right|_{n-\frac{1}{2}} \approx \frac{T_n - T_{n-1}}{\Delta y}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_m \approx \frac{T_{n-1} - 2T_n + T_{n+1}}{\Delta y^2} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_m \approx \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

Balance method:

$$q_x\Big|_{m-\frac{1}{2}} + q_y\Big|_{n-\frac{1}{2}} - q_x\Big|_{m+\frac{1}{2}} - q_y\Big|_{n+\frac{1}{2}} = 0$$

$$q_x\Big|_{m-\frac{1}{2}} = -k\left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2}} A_x \approx -k\cdot\frac{T_{m,n} - T_{m-1,n}}{\Delta x}\cdot(\Delta y\cdot 1)$$

$$q_x\Big|_{m+\frac{1}{2}} = -k\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2}} A_x \approx -k\cdot\frac{T_{m+1,n} - T_{m,n}}{\Delta x}\cdot(\Delta y\cdot 1)$$

$$q_y\Big|_{n-\frac{1}{2}} = -k\left.\frac{\partial T}{\partial y}\right|_{n-\frac{1}{2}} A_y \approx -k\cdot\frac{T_{m,n} - T_{m,n-1}}{\Delta y}\cdot(\Delta x\cdot 1)$$

$$q_y\Big|_{n+\frac{1}{2}} = -k\left.\frac{\partial T}{\partial y}\right|_{n+\frac{1}{2}} A_y \approx -k\cdot\frac{T_{m,n+1} - T_{m,n}}{\Delta y}\cdot(\Delta x\cdot 1)$$

for regular grid $\Delta x = \Delta y$

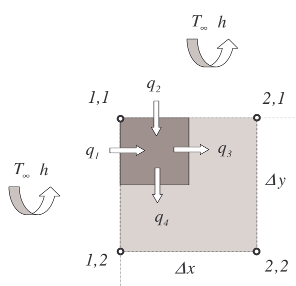
order of approximation $\circ(\Delta x^2)$

Finite-difference equation for interior nodes:

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0 \qquad (4.29)$$

Boundary Nodes:

consider example of convective condition:



$$q_1 + q_2 - q_3 - q_4 = 0$$

$$q_1 = h\cdot(T_\infty - T_{1,1})\cdot\left(\frac{\Delta y}{2}\cdot 1\right) \qquad q_3 = -k\cdot\frac{T_{2,1} - T_{1,1}}{\cancel{\Delta x}}\cdot\left(\frac{\cancel{\Delta y}}{2}\cdot 1\right)$$

$$q_2 = h\cdot(T_\infty - T_{1,1})\cdot\left(\frac{\Delta x}{2}\cdot 1\right) \qquad q_4 = -k\cdot\frac{T_{1,2} - T_{1,1}}{\cancel{\Delta y}}\cdot\left(\frac{\cancel{\Delta x}}{2}\cdot 1\right)$$

$$T_{1,2} + T_{2,1} - 2\left(\frac{h\Delta x}{k} + 1\right)T_{1,1} + \frac{h\Delta x}{k}T_\infty = 0$$

More equations see in Table 4.2 (p.218)

Solution Procedure:

- *Rename $T_{m,n}$ by T_i with a single index i: yields a column-vector* $[\mathbf{T}]$ $(i = 1,2,\dots,N$ *where N is a total number of points where T has to be determined*$)$
- *Rewrite a system of equations in a matrix form* (*system of linear algebraic equations for* T_i): $[\mathbf{A}]\cdot[\mathbf{T}] = [\mathbf{b}]$
- *Solve the matrix equation by a Gaussian elimination or with a help of any numerical solver (see Maple example)*

column-vector of unknowns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$