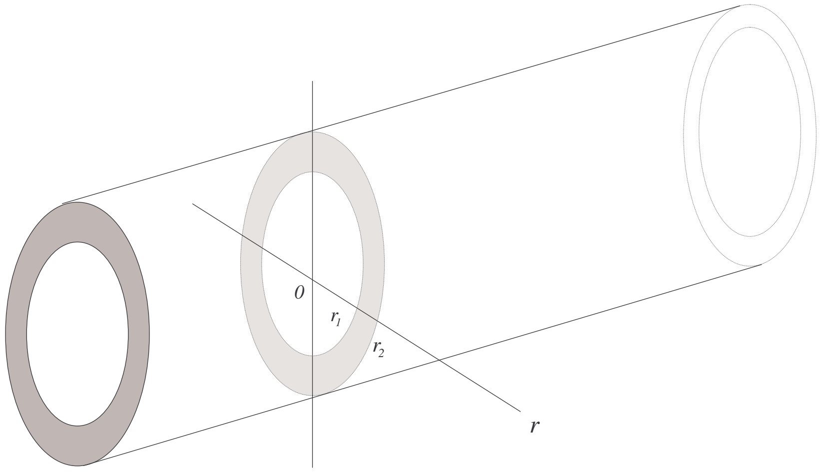
Steady State Conduction in the Cylindrical Wall with Uniform Heat Generation



$\nabla^2 T + F = 0$	$r \in (r_1, r_2)$	$F = \dfrac{\dot{q}}{k} = const$
$r = r_1$	$\left[k\dfrac{\partial T}{\partial r} = h_1\left(T - T_{1,\infty}\right)\right]_{r=r_1}$	**R**
$r = r_2$	$\left[k\dfrac{\partial T}{\partial r} = h_2\left(T_{2,\infty} - T\right)\right]_{r=r_2}$	**R**

There are can be nine combinations of different types of the boundary conditions

Sketch the graph for: $r_1 = 0.2$, $R_2 = 0.5$, $T_{1,\infty} = 50$, $T_{2,\infty} = 10$, $k = 20$, $h_1 = 100$, $h_2 = 200$, $\dot{q} = 500{,}000$

General Solution:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = 0 \qquad r \in (r_1, r_2) \qquad F = \frac{\dot{q}}{k} = const$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{\dot{q}}{k}r$$

$$r\frac{\partial T}{\partial r} = -\frac{\dot{q}}{2k}r^2 + c_1$$

$$\frac{\partial T}{\partial r} = -\frac{\dot{q}}{2k}r + \frac{c_1}{r}$$

Temperature Distribution:

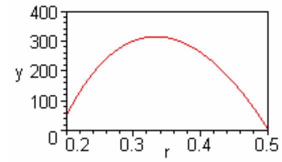
$$T = -\frac{\dot{q}}{4k}r^2 + c_1 \ln r + c_2$$

Heat Flux:

$$q''(r) = \; = \; -k\frac{\partial T}{\partial r} \; = \frac{k\dot{q}}{2k}r - \frac{kc_1}{r}$$

1)

$r = r_1$ $\quad [T = T_1]_{r=r_1}$ $\quad$ **D** $\quad T_1 = -\frac{\dot{q}}{4k} r_1^2 + c_1 \ln r_1 + c_2$



$r = r_2$ $\quad [T = T_2]_{r=r_2}$ $\quad$ **D** $\quad T_2 = -\frac{\dot{q}}{4k} r_2^2 + c_1 \ln r_2 + c_2$

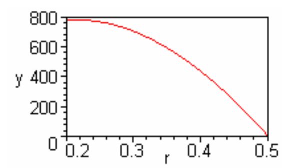
$$T(r) = -\frac{\dot{q}}{4k} r^2 + c_1 \ln r + c_2$$

$$c_1 = \frac{(T_2 - T_1) + \frac{\dot{q}}{4k}\left(r_2^2 - r_1^2\right)}{\ln \frac{r_2}{r_1}}$$

$$c_2 = T_1 + \frac{\dot{q}}{4k} r_1^2 - c_1 \ln r_1$$

2) Bryce Nielsen

$r = r_1$ $\quad \left[-k\frac{\partial T}{\partial r} = q_1''\right]_{r=r_1}$ $\quad$ **N** $\quad \frac{\dot{q}}{2} r_1 - k\frac{c_1}{r_1} = q_1''$



$r = r_2$ $\quad [T = T_2]_{r=r_2}$ $\quad$ **D** $\quad T_2 = -\frac{\dot{q}}{4k} r_2^2 + c_1 \ln r_2 + c_2$

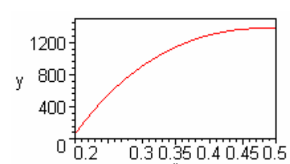
$$T(r) = -\frac{\dot{q}}{4k} r^2 + c_1 \ln r + c_2$$

$$c_1 = \frac{r_1}{k}\left(-q_1'' + \frac{\dot{q}}{2} r_1\right)$$

$$c_2 = T_2 + \frac{\dot{q}}{4k} r_2^2 - c_1 \ln r_2 \qquad = T_2 + \frac{\dot{q}}{4k} r_2^2 + \frac{r_1}{k}\left(q_1'' - \frac{\dot{q}}{2} r_1\right) \ln r_2$$

3) Grant O. Cook III

$r = r_1$ $\quad [T = T_1]_{r=r_1}$ $\quad$ **D** $\quad T_1 = -\frac{\dot{q}}{4k} r_1^2 + c_1 \ln r_1 + c_2$



$r = r_2$ $\quad \left[k\frac{\partial T}{\partial r} = q_2''\right]_{r=r_2}$ $\quad$ **N** $\quad k\frac{\partial T}{\partial r} = q_2'' = -\frac{\dot{q}}{2} r_2 + \frac{c_1 k}{r_2}$

$$T(r) = -\frac{\dot{q}}{4k} r^2 + c_1 \ln r + c_2$$

$$c_1 = \frac{r_2}{k}\left(q_2'' + \frac{\dot{q}}{2} r_2\right)$$

$$= \frac{\dot{q}}{4k}\left(r_1^2 - r^2\right) + \frac{r_2}{k}\left(q_2'' + \frac{\dot{q} r_2}{2}\right) \ln\left(\frac{r}{r_1}\right) + T_1$$

$$c_2 = T_1 + \frac{\dot{q}}{4k} r_1^2 - c_1 \ln r_1$$

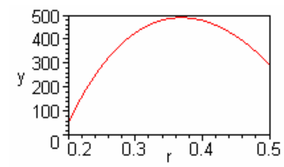
4) Ill-set boundary value problem: N-N

5) Brent Pickett

$r = r_1$ $\quad [T = T_1]_{r=r_1}$ $\quad$ **D**

$$T_1 = -\frac{\dot{q}}{4k} r_1^2 + c_1 \ln r_1 + c_2$$

$$(\ln r_1) c_1 + c_2 = T_1 + \frac{\dot{q}}{4k} r_1^2$$



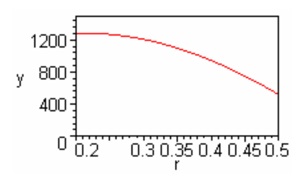
$r = r_2$ $\quad \left[k \frac{\partial T}{\partial r} = h_2 \left(T_{2,\infty} - T\right)\right]_{r=r_2}$ $\quad$ **R**

$$\left(\frac{k}{r_2} + h_2 \ln r_2\right) c_1 + h_2 c_2 = h_2 T_{2,\infty} + \frac{\dot{q} h_2}{4k} r_2^2 + \frac{\dot{q}}{2} r_2$$

$$T(r) = \frac{\dot{q}}{4k}\left(r_1^2 - r^2\right) + \Delta \cdot \ln\left(\frac{r}{r_1}\right) + T_1$$

$$\Delta = \frac{(T_2 - T_1) + \frac{\dot{q}}{4k}\left(r_2^2 - r_1^2\right) + \frac{\dot{q} r_2}{2h_2}}{\frac{k}{h_2 r_2} \ln\left(\frac{r_2}{r_1}\right)}$$

6) Brady Woolford



$r = r_1$ $\quad \left[-k \frac{\partial T}{\partial r} = q_1''\right]_{r=r_1}$ $\quad$ **N**

$$\frac{\dot{q}}{2} r_1 - k \frac{c_1}{r_1} = q_1''$$

$r = r_2$ $\quad \left[k \frac{\partial T}{\partial r} = h_2 \left(T_{2,\infty} - T\right)\right]_{r=r_2}$ $\quad$ **R**

$$\left(\frac{k}{r_2} + h_2 \ln r_2\right) c_1 + h_2 c_2 = h_2 T_{2,\infty} + \frac{\dot{q} h_2}{4k} r_2^2 + \frac{\dot{q}}{2} r_2$$

$$T(r) = -\frac{\dot{q}}{4k} r^2 + c_1 \ln r + c_2$$

$$c_1 = \frac{r_1 \left(\dot{q} r_1 - 2 q_1''\right)}{2k}$$

$$c_2 = T_{2,\infty} + \frac{\dot{q}}{4k} r_2^2 + \frac{\dot{q}}{2h_2} r_2 - \left(\frac{k}{h_2 r_2} + \ln(r_2)\right) \cdot c_1$$

$$c_2 = \frac{1}{4 h_2 r_2 k}\left[2kFr_2^2 - 2k^2 Fr_1^2 + 4 q_1'' r_1 k + 4 h_2 r_2 T_{2,\infty} k + h_2 r_2^2 Fk + 2 h_2 r_2 r_1^2 kF \ln(r_2)\right]$$

7) Daniel Reimann

$r = r_1$ $\qquad \left[k\dfrac{\partial T}{\partial r} = h_1\left(T - T_{1,\infty}\right) \right]_{r=r_1}$ $\qquad$ **R** $\qquad \left(\dfrac{k}{r_1} - h_1 \ln r_1\right)c_1 - h_1 c_2 = -h_1 T_{1,\infty} - \dfrac{\dot{q}h_1}{4k} r_1^2 + \dfrac{\dot{q}}{2} r_1$

$r = r_2$ $\qquad \left[T = T_2 \right]_{r=r_2}$ $\qquad$ **D** $\qquad \left(\ln r_2\right)c_1 + c_2 = T_2 + \dfrac{\dot{q}}{4k} r_2^2$

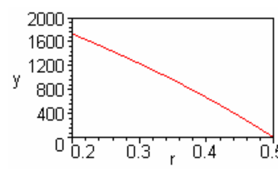
$$T(r) = -\frac{\dot{q}}{4k}r^2 + c_1 \ln r + c_2$$

$$c_1 = \frac{\dfrac{\dot{q}}{4k}\left(r_2^2 - r_1^2\right) + \dfrac{\dot{q}r}{2h_1} + \left(T_2 - T_{1,\infty}\right)}{\dfrac{k}{r_1 h_1} - \ln\left(\dfrac{r_2}{r_1}\right)}$$

$$c_2 = T_2 + \frac{\dot{q}}{4k} r_2^2 - c_1 \ln r_2$$

$$\left(\frac{k}{r_1} - h_1 \ln r_1\right)c_1 - h_1\left(T_2 + \frac{\dot{q}}{4k} r_2^2 - c_1 \ln r_2\right) = -h_1 T_{1,\infty} - \frac{\dot{q}h_1}{4k} r_1^2 + \frac{\dot{q}}{2} r_1$$

$$\left(\frac{k}{r_1} + h_1 \ln\frac{r_2}{r_1}\right)c_1 = h_1\left(T_2 - T_{1,\infty}\right) - \frac{\dot{q}h_1}{4k}\left(r_2^2 - r_1^2\right) + \frac{\dot{q}}{2} r_1$$



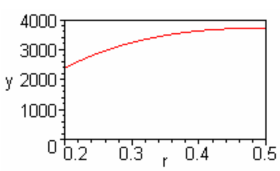
$$c_1 = \frac{\left(T_2 - T_{1,\infty}\right) - \dfrac{\dot{q}}{4k}\left(r_2^2 - r_1^2\right) + \dfrac{\dot{q}}{2h_1} r_1}{\dfrac{k}{r_1 h_1} + \ln\dfrac{r_2}{r_1}}$$

8)

$r = r_1$ $\qquad \left[k\dfrac{\partial T}{\partial r} = h_1\left(T - T_{1,\infty}\right) \right]_{r=r_1}$ $\qquad$ **R** $\qquad k\left(-\dfrac{\dot{q}}{2k} r_1 + c_1\dfrac{1}{r_1}\right) - h_1\left(-\dfrac{\dot{q}}{4k} r_1^2 + c_1 \ln r_1 + c_2\right) = -h_1 T_{1,\infty}$

$r = r_2$ $\qquad \left[k\dfrac{\partial T}{\partial r} = q_2'' \right]_{r=r_2}$ $\qquad$ **N** $\qquad k\dfrac{\partial T}{\partial r} = q_2'' = -\dfrac{\dot{q}}{2} r_2 + \dfrac{c_1 k}{r_2}$

$$c_1 = \frac{r_2}{k}\left(q_2'' + \frac{\dot{q}}{2} r_2\right)$$



$$\left(\frac{k}{r_1} - h_1 \ln r_1\right)c_1 - h_1 c_2 = -h_1 T_{1,\infty} - \frac{\dot{q}h_1}{4k} r_1^2 + \frac{\dot{q}}{2} r_1$$

$$\left(\frac{k}{r_1} - h_1 \ln r_1\right)\left[\frac{r_2}{k}\left(q_2'' + \frac{\dot{q}}{2} r_2\right)\right] - h_1 c_2 = -h_1 T_{1,\infty} - \frac{\dot{q}h_1}{4k} r_1^2 + \frac{\dot{q}}{2} r_1$$

$$c_2 = T_{1,\infty} + \frac{\dot{q}}{4k} r_1^2 - \frac{\dot{q}}{2h_1} r_1 + \frac{r_2}{k}\left(\frac{k}{h_1 r_1} - \ln r_1\right)\left(q_2'' + \frac{\dot{q}}{2} r_2\right)$$

$$T(r) = -\frac{\dot{q}}{4k}r^2 + c_1 \ln r + c_2$$

9)

$$r = r_1 \qquad \left[k\frac{\partial T}{\partial r} = h_1\left(T - T_{1,\infty}\right)\right]_{r=r_1} \qquad \mathbf{R} \qquad k\left(-\frac{\dot q}{2k}r_1 + c_1\frac{1}{r_1}\right) - h_1\left(-\frac{\dot q}{4k}r_1^2 + c_1 \ln r_1 + c_2\right) = -h_1 T_{1,\infty}$$

$$r = r_2 \qquad \left[k\frac{\partial T}{\partial r} = h_2\left(T_{2,\infty} - T\right)\right]_{r=r_2} \qquad \mathbf{R} \qquad k\left(-\frac{\dot q}{2k}r_2 + c_1\frac{1}{r_2}\right) + h_2\left(-\frac{\dot q}{4k}r_2^2 + c_1 \ln r_2 + c_2\right) = h_2 T_{2,\infty}$$

$$\left(\frac{k}{r_1} - h_1 \ln r_1\right)c_1 - h_1 c_2 = -h_1 T_{1,\infty} - \frac{\dot q h_1}{4k}r_1^2 + \frac{\dot q}{2}r_1$$

$$\left(\frac{k}{r_2} + h_2 \ln r_2\right)c_1 + h_2 c_2 = h_2 T_{2,\infty} + \frac{\dot q h_2}{4k}r_2^2 + \frac{\dot q}{2}r_2$$

$$\Delta = \left(\frac{k}{r_1} - h_1 \ln r_1\right)h_2 + \left(\frac{k}{r_2} + h_2 \ln r_2\right)h_1 \quad = k\left(\frac{h_2}{r_1} + \frac{h_1}{r_2}\right) + h_1 h_2 \ln\frac{r_2}{r_1}$$

$$\Delta_1 = \left(-h_1 T_{1,\infty} - \frac{\dot q h_1}{4k}r_1^2 + \frac{\dot q}{2}r_1\right)h_2 + \left(h_2 T_{2,\infty} + \frac{\dot q h_2}{4k}r_2^2 + \frac{\dot q}{2}r_2\right)h_1 \quad = h_1 h_2\left(T_{2,\infty} - T_{1,\infty}\right) + \frac{\dot q h_1 h_2}{4k}\left(r_2^2 - r_1^2\right) + \frac{\dot q}{2}\left(h_2 r_1 + h_1 r_2\right)$$

$$\Delta_2 = \left(\frac{k}{r_1} - h_1 \ln r_1\right)\left(h_2 T_{2,\infty} + \frac{\dot q h_2}{4k}r_2^2 + \frac{\dot q}{2}r_2\right) + \left(\frac{k}{r_2} + h_2 \ln r_2\right)\left(-h_1 T_{1,\infty} - \frac{\dot q h_1}{4k}r_1^2 + \frac{\dot q}{2}r_1\right)$$

$$c_1 = \frac{\Delta_1}{\Delta} = \frac{h_1 h_2\left(T_{2,\infty} - T_{1,\infty}\right) + \frac{\dot q h_1 h_2}{4k}\left(r_2^2 - r_1^2\right) + \frac{\dot q}{2}\left(h_2 r_1 + h_1 r_2\right)}{k\left(\frac{h_2}{r_1} + \frac{h_1}{r_2}\right) + h_1 h_2 \ln\frac{r_2}{r_1}} \quad = \frac{\left(T_{2,\infty} - T_{1,\infty}\right) + \frac{\dot q}{4k}\left(r_2^2 - r_1^2\right) + \frac{\dot q}{2}\left(\frac{r_1}{h_1} + \frac{r_2}{h}\right)}{k\left(\frac{1}{r_1 h_1} + \frac{1}{r_2 h_2}\right) + \ln\frac{r_2}{r_1}}$$

$$c_2 = \frac{\Delta_2}{\Delta} = \frac{\left(\frac{k}{r_1} - h_1 \ln r_1\right)\left(h_2 T_{2,\infty} + \frac{\dot q h_2}{4k}r_2^2 + \frac{\dot q}{2}r_2\right) + \left(\frac{k}{r_2} + h_2 \ln r_2\right)\left(-h_1 T_{1,\infty} - \frac{\dot q h_1}{4k}r_1^2 + \frac{\dot q}{2}r_1\right)}{k\left(\frac{h_2}{r_1} + \frac{h_1}{r_2}\right) + h_1 h_2 \ln\frac{r_2}{r_1}}$$

$$c_2 = \frac{\Delta_2}{\Delta} = \frac{\left(\frac{k}{r_1 h_1} - \ln r_1\right)\left(T_{2,\infty} + \frac{\dot q}{4k}r_2^2 + \frac{\dot q}{2}\frac{r_2}{h_2}\right) + \left(\frac{k}{r_2 h_2} + \ln r_2\right)\left(-T_{1,\infty} - \frac{\dot q}{4k}r_1^2 + \frac{\dot q}{2}\frac{r_1}{h_1}\right)}{k\left(\frac{1}{r_1 h_1} + \frac{1}{r_2 h_2}\right) + \ln\frac{r_2}{r_1}}$$

$$T(r) \qquad = -\frac{\dot q}{4k}r^2 + c_1 \ln r + c_2$$