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Analytical Solution to Steady State and Transient Flow in Ultrahydrophobic Channels

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Abstract
This paper discusses the use of a Finite Fourier Transform Method and an Eigenvalue Expansion Method to calculate analytical solutions for microchannel flow, in which you have alternate regions of no-slip and no-shear. In this paper solutions will be obtained for the steady state scenario. A comparison is made between the solutions achieved using the Finite Fourier Transform Method and the Eigenvalue Expansion Method.

Introduction
Microfluidic transport has recently found importance in a number of emerging technologies in micropower generation, chemical separation processes, integrated microscale heat sinks, cell analysis, and other biomedical diagnostic techniques. As the scale of physical devices decreases, however, surface forces become more important. Classical analysis shows that the required driving pressure to maintain a given flow rate in a circular duct is proportional to the inverse of the channel diameter to the fourth power, $\Delta P \sim D^{-4}$. Thus, at physical scales smaller than 100 μm, generating fluid motion in a microchannel can pose a considerable challenge. Consequently, there has been much recent attention addressing the dynamics of liquid flows through microchannels.

Recently, it has been found that the amount of pressure required to push a fluid though a microchannel can greatly be reduced by the use of micro-riblets coated with a hydrophobic coating. The combination of the micro-riblets and the hydrophobic coating create an ultrahydrophobic surface. When the surface is ultrahydrophobic, the liquid will not penetrate the cavity but will form a meniscus interface that extends slightly into the cavity (as shown below).

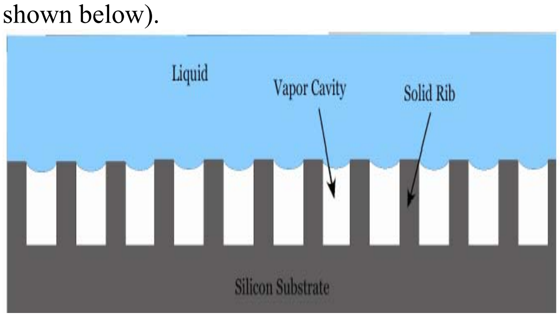


Figure 1. Schematic of liquid flowing over an ultrahydrophobic solid substrate with micro-engineered rigs and cavities.

At the liquid-vapor interface, velocity does not vanish, and thus flow through such a channel can be modeled as a two dimensional microchannel have regions of no-slip and no-shear.

Numerical and experimental data has been collected for the case in which fluid flow is running parallel with the direction of the micro-riblets. Previous work in developing an analytical solution for channel flow with mixed no-slip and no-shear boundary conditions has been limited to a conformal mapping technique that was presented in 1972[1]. In this analytical approach a conformal mapping technique was used to calculate the effects of the no-slip and the no-shear boundary conditions on the flow domain. The paper presented a few results for the case of repeating no-slip, and no-shear boundary conditions, but lacked sufficient detail to make definite conclusions. The purpose of this paper is to show how analytical solutions for the no-slip, no-shear boundary channel flow can be easily obtained by solving the governing equations using one of two methods: an eigenvalue expansion method (EVE) or a Finite Fourier Transform method (FFT). Both of these methods will allow the flow field to be predicted and visualized.

Development of the Analytical Solutions

Assuming constant viscosity, the general Navier-Stokes equations can be reduced to the equation shown below for channel flow in the z direction, where g is the pressure gradient divided by the fluid viscosity.

$$\frac{d^2u}{dy^2} + \frac{d^2u}{d^2x} = g \qquad (1)$$

Both methods obtain analytical solutions for the governing equation given above in equation 1. The governing equation requires prior knowledge of the boundary conditions at x=0, x=L, y=0, and y=M. The bottom and top walls were set as Dirichlet-Dirichlet boundary conditions with the velocity along each of these walls being functions of location along the surface to account for the no-slip and no-shear boundary conditions. The side walls at x=0 and x=M had periodic boundary conditions applied to them. Figure 2 depicts the general setup for the channel used in calculating the analytical solutions in both methods.

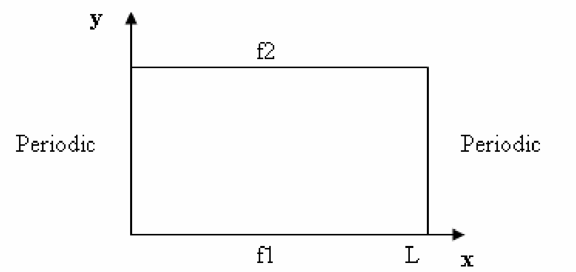


Figure 2 Boundary conditions and geometry used for the analytical solutions with flow into the page.

The solutions for both the FFT method and EVE method are given in the appendix in full detail. A brief summary of how the analytical solution was obtained for both the FFT and EVE methods is given below.

In the FFT method, operational properties are used to transform $\frac{d^2u}{dy^2}$ into a function that is independent of y. The Dirichlet-Dirichlet boundary conditions are also transformed. The transformed form of the equation is given as

$$\frac{d^2u_m}{dx^2} - \mu_m^2 u_m = F_m$$

where F_m is equal to

$$F_m = -G - \sqrt{\frac{2}{L}}\mu_m\left[F_1 + (-1)^{m+1}F_2\right].$$

An assumed solution in the form of

$$u_m = \sum_{n=0}^{\infty}\left[a_{nm}\cos\Lambda_n x + b_{nm}\sin\Lambda_n x\right]$$

was used for the period boundary conditions. By differentiating the assumed solution twice with respect to x, $\frac{d^2u_m}{dx^2}$ is determined and can be substituted into the transformed governing equation. With this substitution the new transformed governing equation is

$$u_m = a_{om} + \sum_{n=0}^{\infty}\left[a_{nm}\cos\Lambda_n x + b_{nm}\sin\Lambda_n x\right].$$

Euler's equations can be used to find the coefficients for periodic boundary conditions. The eigenvalues are given for periodic boundary conditions. The equation is then transformed back into u(x,y) by using the inverse FFT transform.

The EVE method separates the problem into two different simpler problems. The solutions to these problems are determined and then the two solutions are added together to give the complete solution.

$$u(x,y) = u_I(x,y) + u_{II}(x,y)$$

For the $u_I(x,y)$ problem the periodic boundary conditions are set to zero along with any source terms. The governing equation for this first problem is

$$\frac{d^2u_I}{dy^2} + \frac{d^2u_I}{dx^2} = 0$$

and can be solved through separation of variables.

For the $u_{II}(x,y)$ problem f2 and f1 are set to zero and the governing equation is given as

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dx^2} = G.$$

A periodic solution is assumed once again, but in this case the assumed solution will have to be differentiated twice with respect to both x and y. These new equations can be substituted into the part 2 governing equation. The solution is simplified and coefficients are found, which are used in the solution. Both part 1 and part 2 are added together to give the complete solution for the flow field.

In order to apply the no-slip, no-shear boundary conditions along the top and bottom walls of the channel the heaviside function is used. This function allows zero velocity to be applied along the determined no slip fraction and than a value of one along the no-shear fraction. The problems that arise when a value of one is assumed for the no-shear fraction will be addressed in the results section of the paper.

Results and Discussion

A graphical comparison is shown below between the analytical solutions using the FFT method and the EVE method. In this case a 50% no-slip, no shear fraction was used.

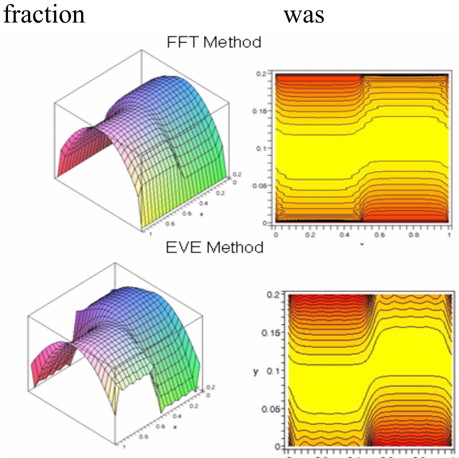


Figure 3 Velocity profiles and contour maps for solutions obtained by both the FFT method and the EVE method

For both cases, very similar graphical results are obtained as would be expected. By integrating over the entire fluid domain and dividing by the length and height of a periodic section of channel the average velocity for that periodic channel section was calculated. It appears that as the amount of no-shear on the bottom and top wall is increased the difference between the average velocities calculated in the FFT method and the EVE increase. This may be a result of the number of terms that are used in the infinite series expansion. The table given below gives average velocity results for the FFT method as the no-shear gap is increased.

Table 1 shows average velocity results calculated for the fluid domain using both the EVE method and the FFT

	EVE	FFT
sheer free gap	averegeV	averageV
0	0.668307	0.668332
50	1.168307	1.16428
75	1.4183	1.41225
85	1.5183	1.5114
95	1.6183	1.61125
99	1.6583	1.6503

For no-slip along both the top and bottom walls the average velocities obtained from the EVE method and the FFT method can be compared to an exact solution. The exact solution gives an average velocity value of 0.668333 m/s for the same dimensions and source term used in the EVE and FFT solutions. The percent error between this value and the values calculated in the EVE and FFT are 0.004% and 0.0002% respectively. So for the case of complete no-slip both the EVE method and FFT method give very accurate results.

An exact solution can also be determined for the case where the bottom wall is no-shear and the top wall is set as no-slip. When the values calculated for the similar situation using the EVE method and FFT method large discrepancies in the average velocities are shown to arise. These discrepancies arise primarily to the use of the heaviside function for the no-shear condition. The heaviside function assigns a value of one to the no-shear region, which is basically setting the velocity along the entire shear region to 1 m/s. This problem can be seen better by comparing the two dimensional velocity profiles obtained using the exact solution and the FFT solution.

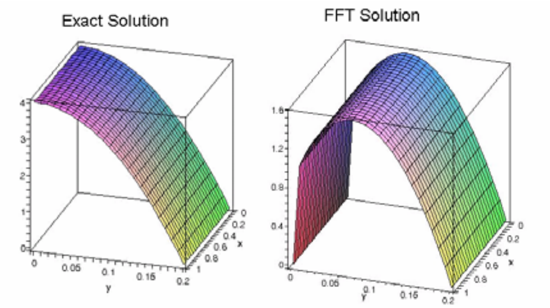


Figure 4 compares the exact solution to the FFT solution for no-slip on top wall no-shear bottom wall

The FFT method does not reach the correct profile because it has the 1 m/s boundary condition at the bottom wall that it is driven to. This same effect is seen as the no-shear to no-slip fraction is increased. The significance of this is that in order to use the FFT or EVE method, as outlined in this paper, to get accurate estimates of the velocity profile in the microchannel the velocity that will exist along the no-shear regions need to be determined. Future work needs to be done in determining these velocities or implementing a different way to apply the boundary conditions instead of using the heaviside function.

Conclusions

The FFT and EVE methods can be used to give quick results to the scenario of fluid flow through a microchannel with mixed no-slip and no-shear boundary conditions. These results are dependent on how the boundary conditions are set, which needs to be investigated further. If an accurate method can be determined for applying the boundary conditions then the results can be assumed to be quit accurate as was shown for the case of no-slip on both walls using the FFT and EVE methods when comparing against the exact solution.

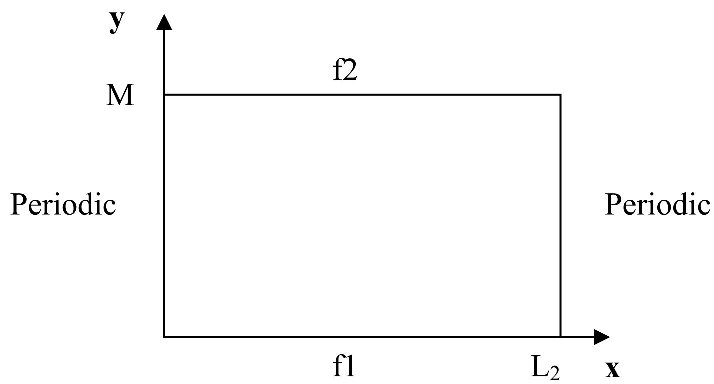
References

[1] Philip, J.R, "*Flow Satisfying Mixed No-Slip and No-Shear Conditions*", Journal of Applied Mathematics and Physics, Vol.23, 1972

APPENDIX I

Eigenfunction Expansion Method for
Periodic Two Dimensional Channel with No-slip and No-shear Boundary Conditions

Initial Setup: Geometry of the cavity is given below. A pressure gradient in the z direction drives the flow in this direction. The no shear boundary condition runs parallel to the flow.



The z momentum equation can be reduced to the form below, where u is the velocity in the z direction.

$$\frac{d^2u}{dy^2}+\frac{d^2u}{dx^2}=G \qquad G=\frac{dp}{\mu_{vis}dx}$$

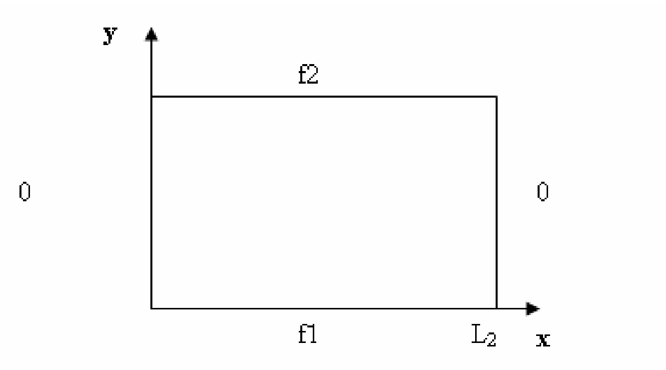
Periodic solution will repeat the solution given in one segment.

$u(x,y)=u(x+L2,y)$

The solution can be obtained by splitting the problem into two parts and solving the partial differential equations for those parts separately and then adding these parts together to get the complete solution.

$u(x,y)=u_I(x,y)+u_{II}(x,y)$

Solution for $u_I(x,y)$**:** Neglect the source term G and solve the PDE



$$\frac{d^2u_I}{dy^2}+\frac{d^2u_I}{dx^2}=0$$

Use separation of variables

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = \frac{Y''}{Y} = \mu$$

Solve for X

$$X'' - \mu X = 0 \qquad \mu = -\lambda^2$$

Second order ODE solution for X

$$X(x) = c_1 \cos \lambda_n x + c_2 \sin \lambda_n x$$

For periodic solution p=L_2/2 and eigenvalues are given as

$$\lambda_n = \frac{2\pi n}{L_2}$$

Substituting the eigenvalues into the solution for X the following is obtained

$$X_n(x) = c_{1,n} \cos\left(\frac{2\pi x}{L_2}\right) + c_{2,n} \sin\left(\frac{2\pi x}{L_2}\right) \text{ with } X_0=1$$

Now, solve the second order ODE for Y using the same technique shown above

$$Y'' + \mu Y = 0$$

$$Y_n(y) = d_{1,n} \cosh\left(\frac{2\pi y}{L_2}\right) + d_{2,n} \sinh\left(\frac{2\pi y}{L_2}\right) \qquad \text{with } Y_0=1$$

Construction of the solution for $u_I(x, y)$ can now be made in the form of an infinite series.

$$u(x, y) = \sum_n c_n X_n(x) Y_n(y)$$

$$u_I(x, y) = b_0 X_0 Y_0 + \sum_n b_n X_n Y_n$$

Substitute and expand terms

$$u_I(x, y) = b_0 c_{1,0} d_{1,M} + \sum_{n=1}^{\infty} b_n \left(c_{1,n} \cos \lambda_n x + c_{1,n} \sin \lambda_n x\right)\left(d_{1,n} \cosh \lambda_n y + d_{1,n} \sinh \lambda_n y\right)$$

$$u_I(x, y) = a_0 + \sum_{n=1}^{\infty} a_{1,n} \cos \lambda_n x \cosh \lambda_n y + a_{2,n} \cos \lambda_n x \sinh \lambda_n y + a_{3,n} \sin \lambda_n x \cosh \lambda_n y + a_{4,n} \sin \lambda_n x \sinh \lambda_n y \qquad \textbf{(1)}$$

Find coefficients by applying boundary conditions $u_I(x,0) = f_1(x)$ and $u_I(x,M) = f_2(x)$ to equation 1.

$$f_1 = a_0 + \sum_{n=1}^{\infty} a_{1,n} \cos \lambda_n x + a_{3,n} \sin \lambda_n x$$

$$a_{0,n} = \frac{1}{L_2} \int_0^{L_2} f_1 dx$$

$$a_{1,n} = \frac{2}{L_2} \int_0^{L_2} f_1 \cos(\lambda_n x) dx$$

$$a_{3,n} = \frac{2}{L_2} \int_0^{L_2} f_1 \sin(\lambda_n x) dx$$

$$f_2 = a_0 + \sum_{n=1}^{\infty} \left[a_{1,n} \cosh \lambda_n M + a_{2,n} \sinh \lambda_n M\right] \cos \lambda_n x + \left[a_{3,n} \cosh \lambda_n M + a_{4,n} \sinh \lambda_n M\right] \sin \lambda_n x$$

$$a_{0,n} = \frac{1}{L_2} \int_0^{L_2} f_2 dx$$

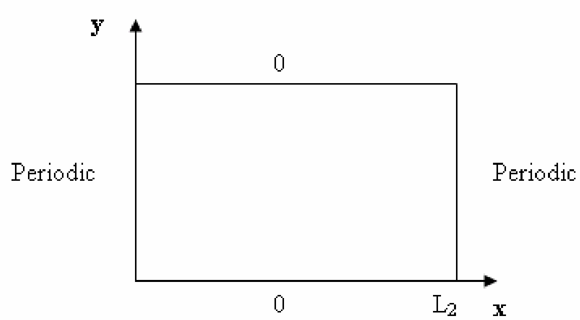
$$a_{2,n} = \frac{\frac{2}{L_2} \int_0^{L_2} f_2 \cos(\lambda_n x) dx - \left[\cosh \lambda_n M\right] a_{1,n}}{\sinh \lambda_n M}$$

$$a_{4,n} = \frac{\frac{2}{L_2} \int_0^{L_2} f_2 \cos(\lambda_n x) dx - \left[\cosh \lambda_n M\right] a_{3,n}}{\sinh \lambda_n M}$$

It should be noted that to get a solution $\int f_1 dx = \int f_2 dx$, if this is not the case then a solution can not be achieved.

Substitute $a_{0,n}$, $a_{1,n}$, $a_{2,n}$, $a_{3,n}$, $a_{4,n}$, into equation 1 to give solution to part I

Solution for $u_{II}(x, y)$: Assume a periodic solution for governing equation which includes the source term and sets f_1,f_2 equal to zero.



$$\frac{d^2 u}{dy^2} + \frac{d^2 u}{dx^2} = G$$

Assumed periodic solution is given below as equation 2.

$$u_{II}(x, y) = \sum_{m=1}^{\infty} a_{o,m} + \sum_{n=1}^{\infty} \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x\right] Y_m(y) \qquad \textbf{(2)}$$

where $Y_m(y) = \sin \mathrm{M}_m y$ which comes from the Sturm-Liouville problem for Dirichlet-Dirichlet boundary conditions. The eigenvalues of $\mathrm{M_m}$ are given as $\frac{m\pi}{M}$. The Λ_n eigenvalues come from the periodic boundary condition and are given by $\frac{2n\pi}{L_2}$.

$$\frac{du_{II}}{dx} = \sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left[-a_{nm}\Lambda_n \sin \Lambda_n x + b_{nm}\Lambda_n \cos \Lambda_n x\right]\right\}Y_m(y)$$
$$\frac{d^2u_{II}}{dx^2} = \sum_{m=1}^{\infty}\left\{\sum_{n=0}^{\infty} -\Lambda_n^2\left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x\right]\right\}Y_m(y)$$
$$\frac{du_{II}}{dy} = \sum_{m=1}^{\infty}\left\{a_{o,m} + \sum_{n=1}^{\infty}\left[-a_{nm}\Lambda_n \sin \Lambda_n x + b_{nm}\Lambda_n \cos \Lambda_n x\right]\right\}\mathrm{M}_m \cos \mathrm{M}_m y$$
$$\frac{d^2u_{II}}{dy^2} = \sum_{m=1}^{\infty}\left\{a_{0,m} + \sum_{n=0}^{\infty} -\Lambda_n^2\left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x\right]\right\}\left(-\mathrm{M}_m^2\right)Y_m(y)$$

Substitute $\frac{d^2u_{II}}{dy^2}$ and $\frac{d^2u_{II}}{dx^2}$ into governing equations and combine terms.

$$\sum_{m=1}^{\infty}\left\{a_{0,m} \sum_{n=1}^{\infty}(\Lambda_n^2 + \mathrm{M}_m^2)\left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x\right]Y_m(y)\right\} = G$$

The coefficients can be determined by setting the left-hand side of the equation above to

$$\frac{2}{M}\int_0^M GY_m dy$$

and solving for the coefficients.

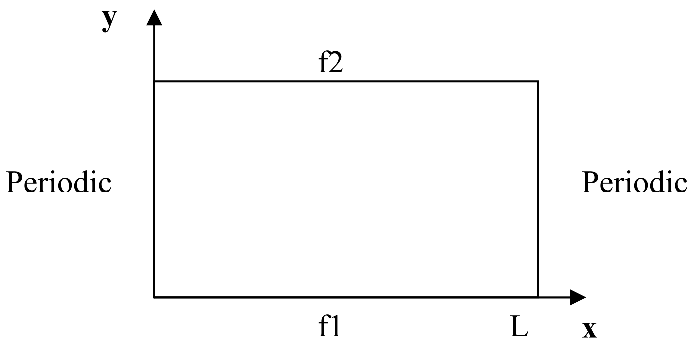
$$a_{om} = \frac{2}{ML_2\mathrm{M}_m^2}\int_0^M\int_0^{L_2} G dx dy$$
$$a_{nm} = \frac{2}{(\Lambda_n^2 + \mathrm{M}_m^2)ML_2}\int_0^{L_2}\int_0^M GY_m \cos \Lambda_n x dy dx$$
$$b_{nm} = \frac{2}{(\Lambda_n^2 + \mathrm{M}_m^2)ML_2}\int_0^{L_2}\int_0^M GY_m \sin \Lambda_n x dy dx$$

The coefficients can now be substituted back into equation 2 to give the solution for $u_{II}(x, y)$.

As stated above the final solution for $u(x, y)$ is obtained when $u_I(x, y)$ and $u_{II}(x, y)$ are added together.

APPENDIX II

Integral Transform Method for periodic channel flow with no-slip no-shear boundary conditions
Setting of the problem:



Z momentum equation reduced to the equation below

$$\frac{d^2u}{dy^2}+\frac{d^2u}{d^2z}=-g \qquad \text{where } g=-\frac{dP}{\mu dz}$$

Apply Finite Fourier Transform to $\frac{d^2u}{dy^2}$ term.

Boundary conditions are Dirichlet-Dirichlet give as:

$$u(x,0)=f_1 \qquad u(x,L)=f_2$$

These boundary conditions give the following Finite Fourier Transform operational properties. (Found in 503 text pg. 145)

$$K_M=\sqrt{\frac{2}{L}}\sin\mu_m y \qquad K_n^{'}=\mu_m\sqrt{\frac{2}{L}}\cos(\mu_m y)$$

Eigenvalues: $\mu_m=\frac{m\pi}{L}$ m=1,2,..

Operational property: $\sqrt{\frac{2}{L}}\mu_m\left[f_1+(-1)^{m+1}f_2\right]-\mu_m^2u_m$

Apply Finite Fourier Transform to governing equation:

$$\frac{d^2u_m}{dx^2}+\sqrt{\frac{2}{L}}\mu_m\left[f_1+(-1)^{m+1}f_2\right]-\mu_m^2u_m=-G$$

Move $\sqrt{\frac{2}{L}}\mu_m\left[f_1+(-1)^{m+1}f_2\right]-\mu_m^2\overline{u_m}$ to right side of equation and combine it with $-G$ and call the function Fm.

$$F_m=-G-\sqrt{\frac{2}{L}}\mu_m\left[f_1+(-1)^{m+1}f_2\right]$$

New form of equation:

$$\frac{d^2u_m}{dx^2}-\mu_m^2u_m=F_m$$

Solution to Periodic boundary condition

Assume the following form for u_m (given on page 210 of 502 text)

$$u_m = \sum_{n=0}^{\infty} \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x \right]$$

Eigenvalues: $\Lambda_n = \frac{2n\pi}{L}$

Differentiate assumed periodic solution twice, simplify, and than substitute into transformed form of govern equation.

$$\frac{du_m}{dx} = \sum_{n=0}^{\infty} \left[-a_{nm}\Lambda_n \sin \Lambda_n x + b_{nm}\Lambda_n \cos \Lambda_n x \right]$$

$$\frac{d^2 u_m}{dx^2} = \sum_{n=0}^{\infty} \left[-a_{nm}\Lambda_n^2 \cos \Lambda_n x - b_{nm}\Lambda_n^2 \sin \Lambda_n x \right]$$

$$\frac{d^2 u_m}{dx^2} = -\sum_{n=0}^{\infty} \Lambda_n^2 \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x \right]$$

$$-\sum_{n=0}^{\infty} \Lambda_n^2 \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x \right] - \mu_m^2 \sum_{n=0}^{\infty} \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x \right] = F_m$$

Combine terms and coefficients:

$$\sum_{n=0}^{\infty} \left[a_{nm}(\Lambda_n^2 + \mu_m^2) \cos \Lambda_n x + b_{nm}(\Lambda_n^2 + \mu_m^2) \sin \Lambda_n x \right] = -F_m$$

Derive equations for the coefficients. The equations are given by Euler's formulas. (or is it by the orthogonality of the periodic equation.)

$$a_{om} = \frac{1}{\mu_m^2 L} \int_0^L \left[-F_m \right] dx$$

$$a_{nm} = \frac{2}{(\Lambda_n^2 + \mu_m^2)L} \int_0^L \left[-F_m \right] \cos(\Lambda_n x) dx$$

$$b_{nm} = \frac{2}{(\Lambda_n^2 + \mu_m^2)L} \int_0^L \left[-F_m \right] \sin(\Lambda_n x) dx$$

New periodic solution for u_m is given below with the coefficients replaced with those derived above.

$$u_m = a_{om} + \sum_{n=0}^{\infty} \left[a_{nm} \cos \Lambda_n x + b_{nm} \sin \Lambda_n x \right]$$

Transform u_m to $u(x, y)$ by applying the Inverse Finite Fourier Transform to give the solution of the velocity profile in a channel.

$$u(x, y) = \sum_{m=1}^{\infty} u_m K_m(y)$$

Do the f1 and f2 functions need to be transformed?