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Fluid Flow between Rotating Concentric Cylinders

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1 Abstract

The Navier Stokes equations govern the motion of fluid in three dimensional space and time. They can be applied to a wide range of fluid flow scenarios and can be solved numerically or, if possible, analytically. The case that we are considering in this paper is a set of concentric cylinders with some viscous fluid in between them. Allowing the rotation of these cylinders will enable us to observe the tangential motion of the fluid throughout time. Similarly, applying a pressure gradient in the axial direction will allow us to solve for the axial motion of the fluid throughout time. The goal of this paper is to analyze this problem using the Navier Stokes equations and solve for the tangential, and axial components of motion simultaneously. The analysis allows us to see the decoupled nature of these two components, which means that the problem is able to be solved analytically, with one component not influencing the other.

2 Nomenclature

- r_i: Inner radius
- r_o: Outer radius
- r: Radius
- V_θ: Tangential velocity
- V_z: Axial velocity
- P: Pressure
- μ: Kinematic viscosity
- ρ: Density
- ν: Dynamic viscosity

3 Introduction

In this paper, we analyze viscous fluid flow in the annular region between concentric rotating cylinders. Specifically where both the fluid and the cylinders are initially at rest, then suddenly, at time t=0, the outer cylinder begins rotating at some rate, ω. Also at t=0, a pressure gradient, $\frac{\partial P}{\partial z}$, is applied in the z direction. The combination of these influences results in a flow in the θ direction as well as in the z direction. Our goal is to analyze these flows and see how they interact.

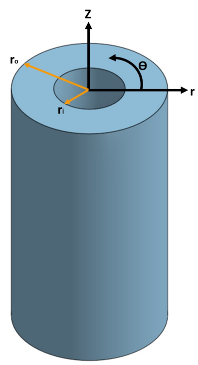

Figure 1. Visualization of radial coordinate system used throughout the report. The inner diameter, r_i, and outer diameter r_o are also indicated.

4 Methodology

We begin by outlining the assumptions necessary to solve the problem. We assumed our fluid is incompressible with a constant viscosity. This reduced the Navier-Stokes Equations to the the following form:

$$\rho(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}) = \frac{-1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)) + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta}]$$

$$\rho(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}) = \frac{-\partial P}{\partial z} + \rho g_z + \mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2}]$$

We then simplified the Navier-Stokes Equations further by making the following assumptions:

1. No flow through walls $\rightarrow V_r = 0$
2. Flow is axisymmetric (invariant in θ) $\rightarrow \frac{\partial}{\partial \theta} = 0$
3. Gravity has a negligible influence $\rightarrow g_\theta = g_z = 0$

Applying these assumptions yields the following partial differential equations:

$$\rho\frac{\partial V_\theta}{\partial t} = \mu\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta))$$

$$\rho\frac{\partial V_z}{\partial t} = -\frac{\partial P}{\partial z} + \mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V_z}{\partial r})]$$

The above PDEs are subject to the following initial and boundary conditions:

$$V_z(r,0) = 0$$
$$V_z(r_i,t) = 0$$
$$V_z(r_o,t) = 0$$

$$V_\theta(r,0) = 0$$
$$V_\theta(r_i,t) = 0$$
$$V_\theta(r_o,t) = r_o\omega(t)$$

Where r_i is the inner radius, r_o is the outer radius, and $\omega(t)$ is the angular velocity of the outer cylinder.

Z-Direction: We begin by solving for flow in the z-direction. To do this, we first find the steady state solution (velocity profile at t=∞). At t=∞, $\frac{\partial}{\partial t} = 0$. Therefore:

$$0 = -\frac{\partial P}{\partial z} + \mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V_z}{\partial r})]$$

This is now an ordinary differential equation which can be solved by separating and integrating twice. The solution is:

$$V_{z,ss} = \frac{\partial P}{\partial Z}\frac{r^2}{4\mu} + C_1 ln(r) + C_2$$

Using boundary conditions, we find the constants C_1 and C_2 to be the following:

$$C_1 = \frac{\partial P}{\partial z}\frac{1}{4\mu}(\frac{r_1^2 - r_2^2}{ln\frac{r_0}{r_i}})$$

$$C_2 = -\frac{\partial P}{\partial z}\frac{r_1^2}{4\mu} - C_1 ln(r_i)$$

Now we must also find the transient solution. We assume the solution to be in the form of:

$$V_z(r,t) = R(r)T(t)$$

Separation of variables yields:

$$\frac{\rho}{\mu}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = \lambda$$

We can rearrange the R equation to the following self adjoint form:

$$[rR']' + [0 + (-\lambda)r]R = 0$$

Thus:

$$r^2R'' + rR' + (\mu^2r^2 - 0)R = 0$$

Where $\mu^2 = -\lambda$. This is the Bessel equation for which the solution is in the following form:

$$R_n(r) = C_{1n}J_0(\mu_n r) + C_{2n}Y_0(\mu_n r)$$

with eigenvalues μ_n being roots of the following characteristic equation:

$$J_0(\mu_n r_i)Y_0(\mu_n r_o) - J_0(\mu_n r_o)Y_0(\mu_n r_i) = 0$$

From our boundary conditions it is possible to find:

$$C_{1n} = \frac{1}{J_0(\mu_n r_o)}$$

$$C_{2n} = \frac{-1}{Y_0(\mu_n r_o)})$$

Thus:

$$R_n(r) = \frac{J_0(\mu_n r)}{J_0(\mu_n r_o)} - \frac{Y_0(\mu_n r)}{Y_0(\mu_n r_o)}$$

We can now solve for $T(t)$. Rearranging the T side of the equation yields:

$$T' + \nu\mu_n^2 T = 0$$

Note: In the above equation, the kinematic viscosity, μ and the density, ρ were combined into the dynamic viscosity, ν, in order to distinguish the kinematic viscosity from the eigenvalue, both represented by μ.

This differential equation can be solved to give

$$T_n(t) = e^{-\nu\mu_n^2 t}$$

Combining these equations back into the form $V_{z,t} = R(r)T(t)$ yields

$$V_{z,t}(r,t) = \sum_n C_n[\frac{J_0(\mu_n r)}{J_0(\mu_n r_o)} - \frac{Y_0(\mu_n r)}{Y_0(\mu_n r_o)}]e^{-\nu\mu_n^2 t}$$

Where:

$$C_n = \frac{\int_{r_i}^{r_o}(u_o - u_{ss})R_n r dr}{|R_n|^2}$$

And thus the final solution can be found by adding together the steady state and transient solutions, $V_{z,ss}$ and $V_{z,t}$. The solution is then:

$$V_z(r,t) = \frac{\partial P}{\partial Z}\frac{r^2}{4\mu} + C_1 ln(r) + C_2 + \sum_n C_n[\frac{J_0(\mu_n r)}{J_0(\mu_n r_o)} - \frac{Y_0(\mu_n r)}{Y_0(\mu_n r_o)}]e^{-\nu\mu_n^2 t}$$

Tangential Direction: With the Z solution solved, now we move on to the θ velocity profile. Starting with our reduced Navier-Stokes equation in the θ direction:

$$\rho\frac{\partial V_\theta}{\partial t} = \mu\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta))$$

We can rearrange the equation to the following form. Again, $\nu = \frac{\mu}{\rho}$

$$\frac{1}{\nu}\frac{\partial V_\theta}{\partial t} = [\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2}]$$

The boundary conditions for the θ equation are time variant, so we will need to apply an integral transform.
In order to find out what this transform should be, we will consider the differential operator:

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2}$$

Subject to boundary conditions of type I-I.

This operator can be put in self-adjoint form with $p = x$

$$Lu = \frac{1}{x}[(xu')' - \frac{u}{x}]$$

Then, we formulate the operator's eigenvalue problem and put it in Sturm-Liouville form:

$$Lu = \lambda y \quad \text{Where} \quad \lambda = -\mu^2$$

$[xy']' + (-\frac{1}{x} + \mu^2 x)y = 0$

We want to find the integral transform for boundary conditions of type I-I, so the eigenvalue problem's boundary conditions become:

$y(x = r_i) = 0$

$y(x = r_o) = 0$

The solutions y_n to this eigenvalue problem can be found with the help of Bessel functions.

$y_n = \frac{J_1(\mu_n x)}{J_1(\mu_n r_o)} - \frac{Y_1(\mu_n x)}{Y_1(\mu_n r_o)}$

Where μ_n are roots of the characteristic equation:

$J_1(\mu_n r_i)Y_1(\mu_n r_o) - J_1(\mu_n r_o)Y_1(\mu_n r_i) = 0$

Now we define an integral transform $\Im$ for the operator Lu with eigenvalues μ_n, eigenfunctions y_n, and weight function $p = x$.
Applying the operator to the V_θ equation yields:

$\frac{\partial \overline{V_{\theta n}}}{\partial t} = -\nu[\mu_n^2 \overline{V_{\theta n}} + y_n'(r_o) r_o * r_o \omega]$

Then apply the Laplace Transform:

$S\hat{\overline{V_{\theta n}}} = -\nu\mu_n^2 \hat{\overline{V_{\theta n}}} - \nu r_o^2 Y_n'(r_o)\hat{\omega}$

$\hat{\overline{V_{\theta n}}} = \frac{-\nu r_o^2 Y_n'(r_2)\hat{\omega}}{S+\nu\mu_n^2}$

Reverse the Laplace Transform:

$\overline{V_{\theta n}} = \int_0^t -\omega(t-\tau)\nu r_o^2 y_n'(r_o) e^{-\nu\mu_n^2\tau} d\tau$

Then reverse the integral transform $\Im$:

$V_\theta(r,t) = \sum_{n=1}^{\infty} \overline{V_{\theta n}} \frac{y_n}{\int_{r_i}^{r_o} (y_n(r))^2 r dr}$

Which is the equation for calculating V_θ.

5 Results

As derived above, we found the solution for flow between two concentric cylinders with a pressure gradient in the z direction by starting with the Navier-Stokes Equations. What was most interesting is we found that the flow's velocity profile in the z-direction and θ direction were independent of each other. In other words, they do not affect one another. So no matter how big or small the pressure gradient, $\frac{\partial P}{\partial z}$, the flow in the θ direction remains the same. Likewise, flow in the z-direction remains constant no matter the velocity of the outer wall. What this means is that the complete, 3 dimensional flow equation is simply:

$V(r,t) = V_\theta + V_z$

This is at least true in the laminar regime. If the flow were to transition to turbulence, our solution would be insufficient. Random turbulent mixing would result in $V_r \neq 0$, invalidating our mathematical model.

Solving the PDE in the z direction with a positive pressure gradient yields the following velocity profiles taken at various points in time.

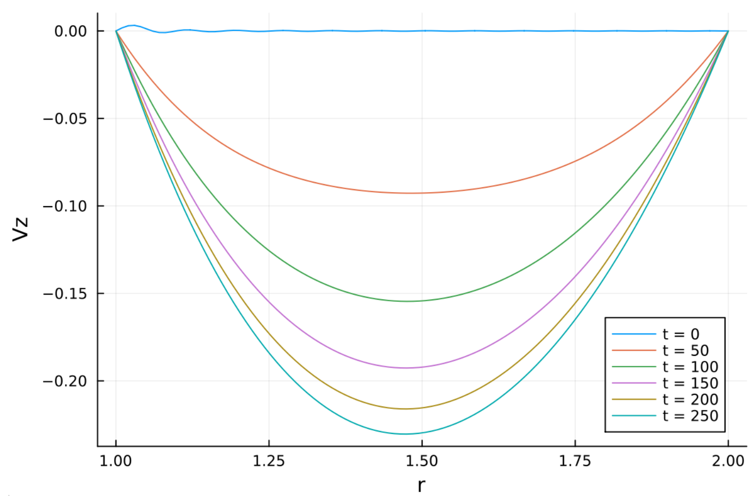

Figure 2. Z component of velocity solution, shown at various instances in time.

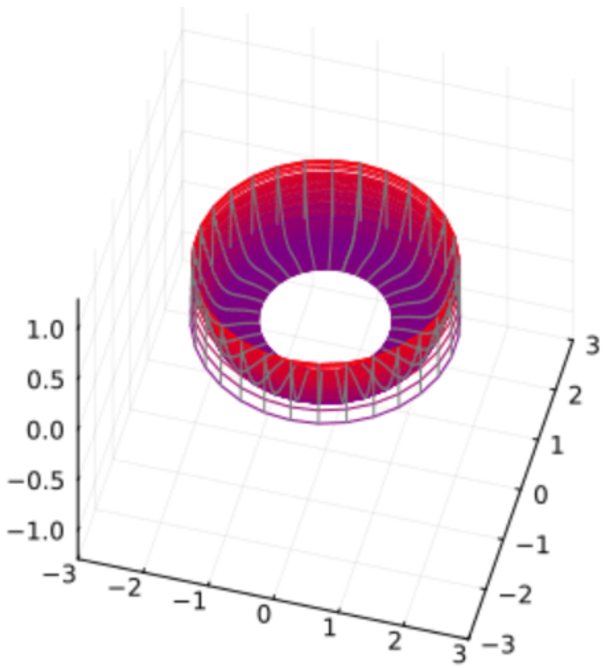

Figure 3. Three dimensional view of the radial velocity after a brief moment.

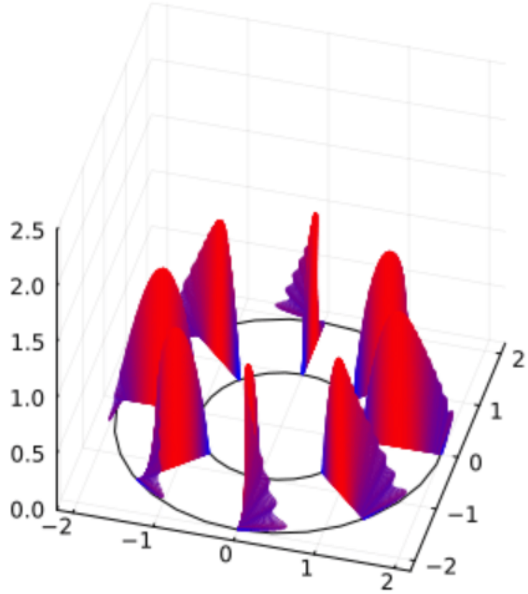

Figure 4. Combined view of the velocity in the z and θ directions. Note that $\frac{\partial P}{\partial z}$ is negative in this view.

6 Conclusion

The most relevant real world application of this problem is perhaps in the lubrication of spinning machinery. For example, both journal and piston rod bearings allow a independent rotation of two cylindrical surfaces while lubricant is pumped.

The most interesting part of our solution is that it demonstrates that V_z is independent of V_θ and vice versa. This means that in such a lubrication application, the pressure needed to pump the lubricant would be the same regardless of the machine's rotation rate so long as the flow stays laminar.

7 Acknowledgments

We would like to thank Dr. Vladimir Soloviev for his assistance in this project.

References

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[2] D Maynes. "Intermediate Fluid Mechanics Notes,". 2021

8 Appendix

Some code as well as several animations were generated for this project (including moving versions of some of the figures). You can find it all on this project's Github repository. The Github repository for this project is available at `https://github.com/tysondanby/505_Project` Feel free to download the repository and change up the parameters to make your own visualizations.