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ANALYSIS OF A STONE SKIPPING ACROSS THE SURFACE OF WATER

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ABSTRACT

This provides a commentary on the methods used to solve and the application of the wave equation. Fourier and Laplace transforms were used to solve the wave equation for a one-dimensional scenario. The surface of a pond was modeled as several points were perturbed, simulating a rock skipping across the surface. From this the resulting wave interactions and behavior at different boundary conditions were observed.

INTRODUCTION

In this study, the surface of a pond was approximated as a one-dimensional line. Several points were agitated to simulate the skipping of a rock across the surface. This problem is set up under the assumption that the thrower is standing on the bank of the pond where x is equal to zero and throwing in the positive x direction. A small portion of the pond was observed where waves are fixed at the bank and free approaching the center of the pond. To capture this behavior, Dirichlet and Robin boundary conditions were selected to solve the wave

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} \tag{1}$$

equation [1]. Each of the S_n terms characterize the amplitude, position and time of the nth rock skip.

$$\frac{\partial^2 U}{\partial x^2} + \sum_n S_n \delta(x - x_n)\delta(t - t_n) = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} \tag{2}$$

S_n is the amplitude of the stone skip. The position and time of the point source are x_n and t_n, respectively.

APPLICATIONS

The wave equation is used across multiple disciplines including fluid dynamics, acoustics, and structural vibrations. It is also used in many industries, such as aerospace, marine, and energy, to name a few. Though our fictitious scenario does not provide the benefit of any useful applications, the methods used, and solution obtained may apply to other situations with similar boundary conditions and moving point sources.

METHODS

The solution and accompanying visuals included two-point sources (e.g. S_1, S_2). However, for simplicity and to accommodating limited space, only the first S_1 term is shown here. If more skips or point sources are desired, add another S_n term multiplied by the appropriate Dirac Delta Functions. The first step in solving this function was to use a **Finite Fourier transform** [2] to convert from space or position to frequency. Equation 3 shows the intermediate step of taking the integral of the left side of the equation times the Eigen Function associated with the boundary condition.

$$S_1\delta(t - t_1)\int_0^L \delta(x - x_1)\sin(\mu\, x)dx \tag{3}$$

Not included are the many Heaviside functions that cancel out after integrating. Going from our initial equation, Equation 2, and performing the described Finite Fourier transform, yields Equation 4.

$$-\mu^2\,\overline{U}(t) + S_1\sin(\mu\, x_1)\,\delta(t - t_1) + \cdots = \frac{1}{v^2}\frac{\partial^2 \overline{U}}{\partial t^2} \tag{4}$$

Following this step, the function is converted from the time domain to the **Laplace** complex frequency domain.

$$-\mu^2\,\boldsymbol{\mathcal{U}} + S_1\sin(\mu\, x_1)\,e^{-t_1 s} + \cdots = \frac{1}{v^2}\boldsymbol{\mathcal{U}}\, s^2 \tag{5}$$

The equation can be rearranged as

$$\boldsymbol{\mathcal{U}} = \frac{v^2 S_1 \sin(\mu\, x_1)\, e^{-t_1 s}}{\mu^2 v^2 + s^2} \tag{6}$$

to isolate $\boldsymbol{\mathcal{U}}$. Now we can perform inverse Fourier and inverse Laplace transforms where H is the Heaviside Function.

$$\bar{U} = S_1 \sin(\mu\, x_1)\, H(t - t_1) \sin(\mu v(t - t_1)) \frac{1}{\mu v} \tag{7}$$

The $\bar{U}$ from Equation 7 is the $\bar{U}$ in the final solution, Equation 8.

$$U(x,t) = \sum_{n}^{\infty} \bar{U} \frac{\sin(\mu\, x)}{\frac{L}{2} - \frac{\sin(2\mu L)}{4\mu}} \tag{8}$$

RESULTS

From the boundary conditions and derived equation, a solution to the wave equation was obtained. Graphical representations have been created to observe the waves' interactions with each other. The surface of the pond following the first skip of the rock is shown in Figure 1. The interactions are most noticeable near x=2.5, the location where the rock hits the surface. Figure 2 shows the surface following the second skip of the rock. In this case, ripples in the surface extend further and interactions with the boundary condition at x=10 are observed.

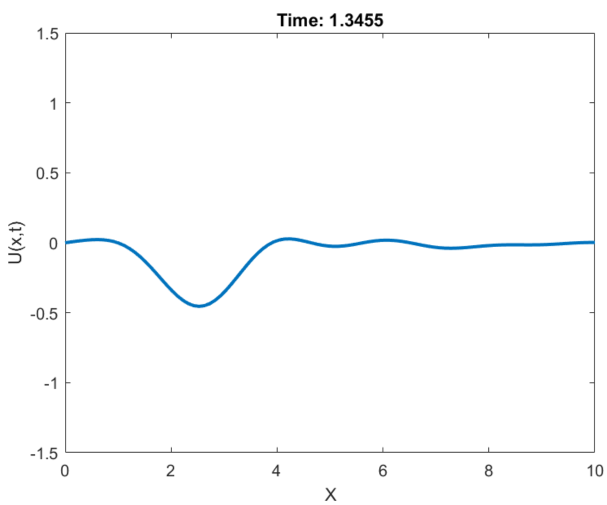

Figure 1. The surface of the pond immediately after the first skip.

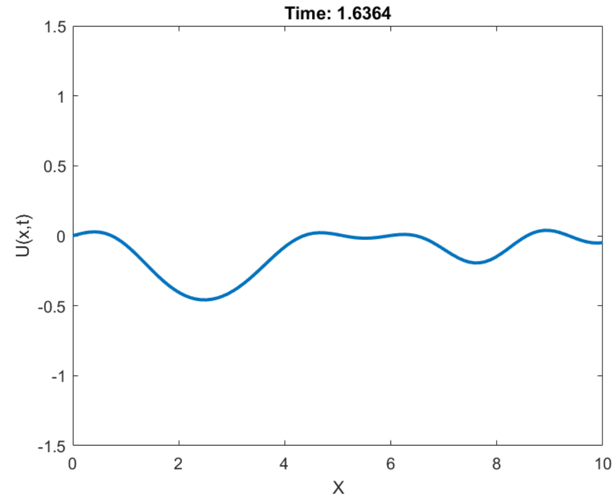

Figure 2. The surface of the pond immediately after the second skip.

Because of the low fidelity of this model, the highest physical meaning can only be obtained very shortly after the second stone skip. No dampening was considered for the purpose of observing the waves interactions with each other and the boundaries.

CONCLUSIONS

This model shows the behavior of the waves interacting with each other and the boundaries. The assumptions made to observe this scenario are only valid for waves propagating near each other and early interactions with the boundaries. Dampening was not included in the model, and it is only physically accurate for a short period of time following perturbations.

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REFERENCES

[1] Soloviev, V., 2021, "Integrated Engineering Mathematics" BYU, pp. 673.
[2] Soloviev, V., 2021, "Integrated Engineering Mathematics" BYU, pp. 758-759.

APPENDIX

Code used to create animation:

```
% Constants
L = 10;          % Length of the domain
t_max = 10;      % Maximum time
N = 100;         % Number of spatial points
```



```
M = 275;        % Number of time steps

v = 1;          % Velocity

S1 = .75;       % Source 1 amplitude
x1 = L/4;       % Source 1 position
t1 = t_max*0.05;         % Source 1 time

S2 = 0.5;       % Source 2 amplitude
x2 = 3*L/4;     % Source 2 position
t2 = t_max*0.13;         % Source 2 time

H2 = 1;       % Parameter in the
% transcendental equation
mu_n = 10;
% Find up to the nth positive root of the
% transcendental equation
mu_roots = find_roots_up_to_n(mu_n, L, H2);

% Spatial grid
x = linspace(0, L, N);

% Preallocate the matrix for storing the wave
% at different time steps
U = zeros(N, M);

% Time loop
for k = 1:M
    t = (k-1) * t_max / M;  % Current time
    % Evaluate the wave equation at each
%spatial point
    for i = 1:N
        for j = 1:mu_n
            ubar = v/mu_roots(j)*-
S1*heaviside(t-
t1)*sin(mu_roots(j)*x1)*sin(mu_roots(j)*v*(t-
t1)) + v/mu_roots(j)*-S2*heaviside(t-
t2)*sin(mu_roots(j)*x2)*sin(mu_roots(j)*v*(t-
t2));
            U(i, k, j) = ubar *
(sin(mu_roots(j)*x(i))/(L/2-
sin(mu_roots(j)*L)/(4*mu_roots(j))));
            % Unew = squeeze(sum(U,3));

        end
    end
end
Unew = sum(U,3);

% Create a figure and axis for animation
figure;
axis tight manual;
xlabel('X');
ylabel('U(x,t)');
% title('1D Wave Equation Animation');

% Update the plot at each time step
for k = 1:M

    plot(x, Unew(:, k), 'LineWidth', 2);
    ylim([-1.5, 1.5]);  % Set the y-axis
%limits as needed
    title(['Time: ' num2str((k-1) * t_max /
M)]);
    xlabel('X');
    ylabel('U(x,t)');
    drawnow;
end

% Function to find up to the nth positive root
%of the equation
function mu_roots = find_roots_up_to_n(n, L,
H2)
    % Define the equation
    eq = @(muu) muu * cos(muu * L) + H2 *
sin(muu * L);

    % Initialize an array to store the roots
    mu_roots = zeros(1, n);

    % Find the first n positive roots using
%fzero
    for i = 1:n
        % Use fzero to find the root in the i-
%th interval
        initial_guess = (i - 0.5) * pi / L;  %
%Start with an initial guess in the middle of
%the interval
        mu_roots(i) = fzero(eq,
initial_guess);
    end
end
```

