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USING AN ANALYTICAL SOLUTION OF 2D HEAT DIFFUSION EQUATION TO EVALUATE THE EFFECTS OF WELD SPEED

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ABSTRACT

Friction Stir Welding (FSW) is a solid-state joining process used widely in the automotive and aerospace industries. Here, a 2-D heat diffusion model was used to determine the effects of weld speed and thermal properties on the thermal history of the welded material. It was found that as weld speed went up the heat wake decreased in width and the average temperature went down.

NOMENCLATURE

T_∞= Ambient Temperature
v_c = Crossover Weld Speed
α = Thermal Diffusivity

INTRODUCTION

The intent of this work is to develop a model for heat diffusion through an aluminum plate undergoing friction stir welding (FSW). The resulting conduction through the plate can be used to see the effects of weld speed on the heat wake and average temperature.

When using FSW, the peak temperature and the cooling rate of the material are the primary factors that affect the properties of the weld, because time and temperature drive the metallurgical response of the welded metal [1]. Therefore, it is important to know how changes to the FSW process affect the temperature history.

The developed model was used to determine the crossover weld speed when the FSW tool starts moving faster than the heat in the plate can propagate.

METHODS

To model the aluminum plate, the 2D Heat Equation was used with an included source term to model the heat input from the welding process:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + S(x,y,t) = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

To adequately model the heat transfer from the aluminum plate, the following boundary conditions were chosen:

$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{x=L} = 0$$

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = 0, \qquad \left[\frac{\partial u}{\partial y}\right]_{y=M} = 0$$

$$S(x,y,t) = \phi_0 \cdot \delta(x - vt) \cdot \delta\left(y - \frac{M}{2}\right)$$

The boundaries of the aluminum were modeled by insulated faces in both the x and y direction. These boundary conditions correspond with Neumann-Neumann boundaries in the x-direction and y-directions. The source term was then applied to the centerline of the plate and moved the x-direction at a given speed.

The initial conditions used corresponded with the following equations where 0 is considered the ambient temperature as seen below:

$$u|_{t=0} = 0$$

$$u(x,y,t) = T(x,y,t) - T_\infty$$

By using the integral transform method, the model described can be solved for the function $u(x, y, t)$ where u represents the temperature of the aluminum block. A detailed description of the solution can be found in the Appendix along with code to show how the Fourier and Laplace transforms were created.

RESULTS

The resulting equation gave the temperature of the aluminum block undergoing friction stir welding (u) as a function of position (x, y, z) and time (t). All components can be found in detail within the Appendix.

$$u(x, y, t) = \sum_n \sum_m \frac{X_n(x)}{\|X_n(x)\|^2} \frac{Y_m(y)}{\|Y_m(y)\|^2} \cdot \bar{\bar{u}}_{n,m}(t)$$

In the example aluminum block chosen, a sharp temperature rise was seen at the block's edge that moves along the middle of the block at the same speed as the friction stir weld. As the heat input from the welding process moves along in the x-direction, it was seen that the heat diffused faster than the welder moved leading to higher temperatures at the welding location. There was also a trailing temperature increase behind the weld that flattened out the farther away it was from the source of the heat input.

Four different weld speeds were chosen for evaluation with the thermal properties of Al 7075-T6: 50 mm/min, 100 mm/min, 300 mm/min, and 500 mm/min. Each weld speed was evaluated when it reached the middle of the plate in the x direction.

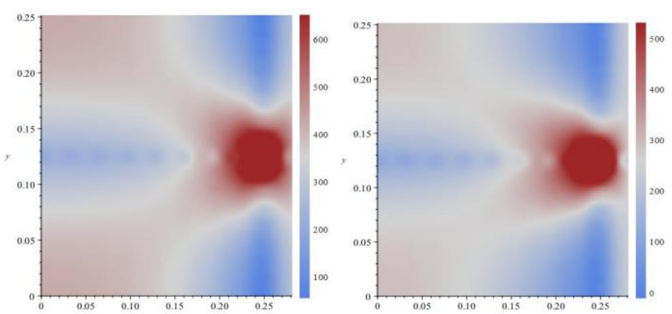


***Figure 1**: Temperature distribution of 50 mm/min (left) and 100 mm/min (right) weld speeds.*

Because of this choice, the average temperatures of the plates for each welding speed were different, but the overall characteristics of the temperature gradient could still be seen. The slower weld speed of 50 mm/min has a higher average temperature and a wider heat wake than the 100 mm/min weld speed. This makes sense because the slower weld speed takes longer to complete the weld which will give more time for heat generation. The wider heat wake can be explained by the additional time for heat diffusion.

Similar trends are seen in **Figure 2**. Additionally, it was observed that the heat wake became narrower as welding speed increased which does not allow the heat to propagate to edge of the plate as compared to **Figure 1**. The temperature in front of the source term is also lower than in **Figure 1** potentially due to the boundary conditions chosen and number of terms used in the summation. It was concluded that the crossover speed was $100\,\frac{mm}{min} < v_c < 300\,\frac{mm}{min}$.

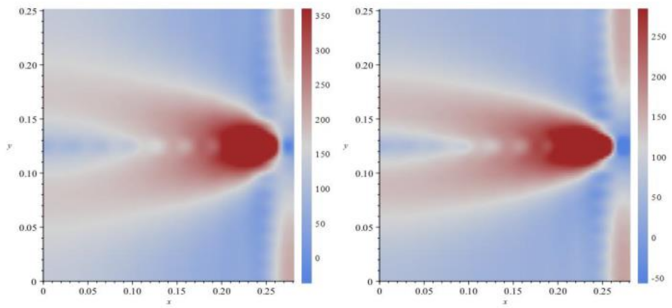


***Figure 2**: Temperature distribution of 300 mm/min (left) and 500 mm/min (right) weld speeds.*

More analysis could be done with the existing model, but due to the model uncertainty, a more specific crossover velocity was not determined. 30 terms were used for the creation of this model, and it is unclear if this was enough terms for a precise determination of the crossover speed. Additionally, the exact definition of v_c would need to be something relatively simple to evaluate in a model and real life to be practical. Another factor to be considered is the weld path. In this model, it was assumed to be straight, but other welds paths need to be taken into consideration.

CONCLUSIONS

As mentioned previously, the developed model was created in order to find the crossover speed at which the FSW tool moves faster than the heat propagates allowing for more control over the thermal history of the plate which ultimately drives post-weld mechanical properties.

It was found that $100\frac{mm}{min} < v_c < 300\frac{mm}{min}$ for Al 7075-T6.

Further work can be done to improve the model. A 3D heat diffusion model with different boundary conditions would more accurately represent true welding conditions. This model could also be compared to real welds and/or a validated FEA model.

ACKNOWLEDGMENTS

Put acknowledgments here.

REFERENCES

[1] B.J. Stringham, T.W. Nelson, C.D. Sorensen, Non-dimensional modeling of the effects of weld parameters on peak temperature and cooling rate in friction stir welding, Journal of Materials Processing Technology 255 (2018) 816-830.

APPENDIX

Denote $u(x,y,t) = T(x,y,t) - T_\infty$

Thermal diffusivity: $\alpha = \frac{\kappa}{\rho c_p}$

$H = \dfrac{h}{\kappa}$

Heat Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + S(x,y,t) = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

Boundary Conditions:

$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{x=L} = 0$$

$$\left[\frac{\partial u}{\partial y}\right]_{y=0} = 0, \qquad \left[\frac{\partial u}{\partial y}\right]_{y=M} = 0$$

$$\phi(x,y,t) = \phi_0 \cdot \delta(x - vt) \cdot \delta\left(y - \frac{M}{2}\right)$$

Initial Condition:

$$u|_{t=0} = 0$$

$X'' - \zeta X = 0$	
Boundary Conditions	**N** $u'(0) = f_0 = 0$ **N** $u'(L) = f_L = 0$
Eigenvalues γ_n, β_m	$\gamma_n = \dfrac{n\pi}{L}$ $n = 0,1,2,\ldots$
Eigenfunctions	$cos(\gamma_n x)$
Norm	$L, n = 0\,;\ \dfrac{L}{2}\ n = 1,2,\ldots$
Operational Property $\int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right) X_n(x)dx$	$f_L X_n(L) - f_0 X_n(0) - \gamma_n^2 \bar{u}_n$

$Y'' - \eta Y = 0$	
Boundary Conditions	**N** $u'(0) = f_0 = 0$ **N** $u'(M) = f_M = 0$
Eigenvalues γ_n, β_m,	$\beta_m = \dfrac{m\pi}{M}$ $n = 0,1,2,\ldots$
Eigenfunctions	$cos(\beta_m y)$
Norm	$M, n = 0\,;\ \dfrac{M}{2}\ n = 1,2,\ldots$
Operational Property $\int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right) X_n(x)dx$	$f_M Y_m(M) - f_0 Y_m(0) - \beta_m^2 \bar{u}_m$

Integral Transform x:

$$-\gamma_n^2\bar{u}_n(y,t)+\frac{\partial^2\bar{u}_n(y,t)}{\partial y^2}+\bar{S}(y,t)=\frac{1}{\alpha}\frac{\partial\bar{u}_n(y,t)}{\partial t}$$

$$\bar{S}(y,t)=\phi_0\cdot\Im_x\{\delta(x-vt)\}\cdot\delta\left(y-\frac{M}{2}\right)$$

Integral Transform y: (3 line to 2 lines)

$$-\gamma_n^2\bar{\bar{u}}_{n,m}(t)-\beta_m^2\bar{\bar{u}}_{n,m}(t)+\bar{\bar{\hat{S}}}(t)=\frac{1}{\alpha}\frac{\partial\bar{\bar{u}}_{n,m}(t)}{\partial t}$$

$$\bar{S}(t)=\phi_0\cdot\Im_x\{\delta(x-vt)\}\cdot\Im_y\left\{\delta\left(y-\frac{M}{2}\right)\right\}$$

Laplace Transform: 3 lines to 2

$$-\gamma_n^2\hat{\bar{\bar{u}}}_{n,m}(s)-\beta_m^2\hat{\bar{\bar{u}}}_{n,m}(s)+\hat{\bar{\bar{S}}}(s)=\frac{1}{\alpha}s\hat{\bar{\bar{u}}}_{n,m}(s)$$

$$\hat{\bar{\bar{S}}}(s)=\phi_0\cdot\mathcal{L}\{\Im_x\{\delta(x-vt)\}\}\cdot\Im_y\left\{\delta\left(y-\frac{M}{2}\right)\right\}$$

Rearrange Terms:

$$\hat{\bar{\bar{S}}}(s)=\frac{1}{\alpha}s\hat{\bar{\bar{u}}}_{n,m}(s)+\gamma_n^2\hat{\bar{\bar{u}}}_{n,m}(s)+\beta_m^2\hat{\bar{\bar{u}}}_{n,m}(s)$$

$$\alpha\hat{\bar{\bar{S}}}(s)=(s+\alpha\gamma_n^2+\alpha\beta_m^2)\,\hat{\bar{\bar{u}}}_{n,m}(s)$$

$$\hat{\bar{\bar{u}}}_{n,m,k}(s)=\alpha\hat{\bar{\bar{S}}}(s)\frac{1}{(s+\alpha\gamma_n^2+\alpha\beta_m^2)}$$

$$\hat{\bar{\bar{S}}}(s)=\phi_0\cdot\mathcal{L}\{\Im_x\{\delta(x-vt)\}\}\cdot\Im_y\left\{\delta\left(y-\frac{M}{2}\right)\right\}$$

Inverse Laplace Transform:

$$\bar{\bar{u}}_{n,m}(t)=\alpha\bar{\bar{S}}(t)*\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha\gamma_n^2+\alpha\beta_m^2)}\right\}$$

$$\bar{\bar{S}}(t)=\phi_0\cdot\Im_x\{\delta(x-vt)\}\cdot\Im_y\left\{\delta\left(y-\frac{M}{2}\right)\right\}$$

$$where * is\ the\ convolution$$

Inverse Integral Transforms:

$$u(x,y,t) = \sum_{n=0}\sum_{m=0} \frac{X_n(x)}{\|X_n(x)\|^2}\frac{Y_m(y)}{\|Y_m(y)\|^2} \cdot \bar{\bar{\bar{u}}}_{n,m}(t)$$

$$\bar{\bar{u}}_{n,m}(t) = \alpha\bar{\bar{S}}(t) * \mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha\gamma_n^2+\alpha\beta_m^2)}\right\}$$

$$\bar{\bar{S}}(t) = \phi_0 \cdot \mathfrak{I}_x\{\delta(x-vt)\} \cdot \mathfrak{I}_y\left\{\delta\left(y-\frac{M}{2}\right)\right\}$$

$$where * is\ the\ convolution$$

$$note\ that\ at\ n=0, \quad X_0(x)=1, \quad \|X_0(x)\|^2 = L, \quad Y_0(y)=1, \quad \|Y_0(y)\|^2 = M$$

Physical Properties	Value	Units
κ (Thermal Conductivity)	130	$\frac{W}{m \cdot K}$
c_p (Heat Capacity)	960	$\frac{J}{kg \cdot K}$
ρ (Density)	2810	$\frac{kg}{m^3}$
ϕ_0 (Heat Flux)	1500	$\frac{W}{m^2}$
L	0.5	m
M	0.25	m

restart;

Constants:
local gamma :**local** beta :

X Eigenvalue/Eigenfunction (II-II):
$\gamma[0] := 0;$

$$\gamma_0 := 0 \tag{1}$$

$X[0] := 1;$

$$X_0 := 1 \tag{2}$$

$normsqX0 := L;$

$$normsqX0 := L \tag{3}$$

$\gamma[n] := \frac{n \cdot \pi}{L};$

$$\gamma_n := \frac{n\,\pi}{L} \tag{4}$$

$X[n] := \cos(\gamma[n] \cdot x);$

$$X_n := \cos\left(\frac{n\,\pi\,x}{L}\right) \tag{5}$$

$normsqX := \frac{L}{2};$

$$normsqX := \frac{L}{2} \tag{6}$$

Y Eigenvalue/Eigenfunction (II-II):
$\beta[0] := 0;$

$$\beta_0 := 0 \tag{7}$$

$Y[0] := 1;$

$$Y_0 := 1 \tag{8}$$

$normsqY0 := M;$

$$normsqY0 := M \tag{9}$$

$\beta[m] := \frac{m \cdot \pi}{M};$

$$\beta_m := \frac{m\,\pi}{M} \tag{10}$$

$Y[m] := \cos(\beta[m] \cdot y);$

$$Y_m := \cos\left(\frac{m\,\pi\,y}{M}\right) \tag{11}$$

$normsqY := \frac{M}{2};$

$$normsqY := \frac{M}{2} \tag{12}$$

Transforms:

$Fx := int(X[n]\cdot \mathrm{Dirac}(x - v\cdot t), x = 0 .. L);$

$$Fx := (\mathrm{Heaviside}(-v\,t + L) - \mathrm{Heaviside}(-v\,t))\cos\left(\frac{n\,\pi\,v\,t}{L}\right) \tag{13}$$

$Fy := int\left(Y[m]\cdot \mathrm{Dirac}\left(y - \frac{M}{2}\right), y = 0 .. M\right);$

$$Fy := (2\,\mathrm{Heaviside}(M) - 1)\cos\left(\frac{m\,\pi}{2}\right) \tag{14}$$

$S := \phi 0\cdot Fx\cdot Fy;$

$$S := \phi 0\,(\mathrm{Heaviside}(-v\,t + L) - \mathrm{Heaviside}(-v\,t))\cos\left(\frac{n\,\pi\,v\,t}{L}\right)(2\,\mathrm{Heaviside}(M) - 1)\cos\left(\frac{m\,\pi}{2}\right) \tag{15}$$

$with(inttrans):$
$assume(0 < a)$
$S := laplace(S, t, s);$

$$\begin{aligned} S := {} & 2\,\mathcal{L}\left(\cos\left(\frac{n\,\pi\,v\,t}{L}\right)\mathrm{Heaviside}(-t\,v + L), t, s\right)\cos\left(\frac{m\,\pi}{2}\right)\mathrm{Heaviside}(M)\,\phi 0 \\ & - 2\,\mathcal{L}\left(\cos\left(\frac{n\,\pi\,v\,t}{L}\right)\mathrm{Heaviside}(-v\,t), t, s\right)\cos\left(\frac{m\,\pi}{2}\right)\mathrm{Heaviside}(M)\,\phi 0 \\ & - \mathcal{L}\left(\cos\left(\frac{n\,\pi\,v\,t}{L}\right)\mathrm{Heaviside}(-t\,v + L), t, s\right)\cos\left(\frac{m\,\pi}{2}\right)\phi 0 \\ & + \mathcal{L}\left(\cos\left(\frac{n\,\pi\,v\,t}{L}\right)\mathrm{Heaviside}(-v\,t), t, s\right)\cos\left(\frac{m\,\pi}{2}\right)\phi 0 \end{aligned} \tag{16}$$

$U := invlaplace\left(\frac{\alpha\cdot S}{s + \alpha\cdot\gamma[n]^2 + \alpha\cdot\beta[m]^2}, s, t\right);$

$$\begin{aligned} U := {} & \Bigg(\Bigg(-\pi\Bigg(\lim_{\_U1 \to 0+}\Bigg(\Bigg(\pi\,(-\mathrm{Heaviside}(-\_U1\,v) + \mathrm{Heaviside}(-v\,t))\,e^{-t\,\alpha\,\pi^2\left(\frac{m^2}{M^2} + \frac{n^2}{L^2}\right)}(L^2\,m^2 \\ & + M^2\,n^2)\,\mathrm{signum}(v)\,\alpha - \left(L\,M^2\sin\left(\frac{n\,\pi\,v\,\_U1}{L}\right)n\,v + \alpha\cos\left(\frac{n\,\pi\,v\,\_U1}{L}\right)\pi\,(L^2\,m^2 \right. \\ & \left. + M^2\,n^2)\right)e^{-\frac{\pi^2\,\alpha\,(L^2\,m^2 + M^2\,n^2)\,(t - \_U1)}{M^2\,L^2}}\mathrm{Heaviside}(-\_U1\,v) + \Bigg(n\,v\,M^2\,L\sin\left(\frac{n\,\pi\,v\,t}{L}\right) \\ & + \pi\,\alpha\,(L^2\,m^2 + M^2\,n^2)\cos\left(\frac{n\,\pi\,v\,t}{L}\right)\Bigg)\mathrm{Heaviside}(-v\,t)\Bigg)L^2\,M^2\Bigg) \Bigg/ \Big(\pi\,(L^4\,\pi^2\,\alpha^2\,m^4 \end{aligned} \tag{17}$$

$$+ 2L^2M^2\pi^2\alpha^2m^2n^2 + M^4\pi^2\alpha^2n^4 + n^2v^2M^4L^2))\Big) (L^4\pi^2\alpha^2m^4 + 2L^2M^2\pi^2\alpha^2m^2n^2$$

$$+ M^4\pi^2\alpha^2n^4 + n^2v^2M^4L^2) + \Bigg(-\Big(\pi\alpha(L^2m^2 + M^2n^2)\,\mathrm{e}^{-t\alpha\pi^2\left(\frac{m^2}{M^2}+\frac{n^2}{L^2}\right)} + (\pi\alpha(L^2m^2$$

$$+ M^2n^2)\cos(n\pi v) + L\sin(n\pi v)M^2n|v|)\,\mathrm{e}^{-\frac{\pi^2\alpha(L^2m^2+M^2n^2)(-\mathrm{signum}(v)L+t)}{M^2L^2}}\,\mathrm{signum}(v)\Big)$$

$$\mathrm{Heaviside}(L) + \Bigg(n\,v\,M^2L\sin\left(\frac{n\pi v t}{L}\right) + \pi\alpha(L^2m^2 + M^2n^2)\cos\left(\frac{n\pi v t}{L}\right)$$

$$+ (\pi\alpha(L^2m^2 + M^2n^2)\cos(n\pi v)$$

$$+ L\sin(n\pi v)M^2n|v|)\,\mathrm{e}^{-\frac{\pi^2\alpha(L^2m^2+M^2n^2)(-\mathrm{signum}(v)L+t)}{M^2L^2}}\,\mathrm{signum}(v)\Bigg)\,\mathrm{Heaviside}(-vt + L)\Bigg)$$

$$L^2M^2\Bigg)\cos\left(\frac{m\pi}{2}\right)(2\,\mathrm{Heaviside}(M) - 1)\,\alpha\,\phi 0\Bigg) \Bigg/ \Big(\pi\,(L^4\pi^2\alpha^2m^4 + 2L^2M^2\pi^2\alpha^2m^2n^2$$

$$+ M^4\pi^2\alpha^2n^4 + n^2v^2M^4L^2)\Big)$$

$n := 0 : m := 0 :$

$$U0 := invlaplace\left(\frac{\alpha\cdot S}{s + \alpha\cdot\gamma[n]^2 + \alpha\cdot\beta[m]^2}, s, t\right);$$

$$U0 := \Big(-\mathrm{Heaviside}(L)\,L + \mathrm{Heaviside}(-vt + L)\,(t + L) - \Big(\lim_{\_U1\to 0+}(-\_U1\,\mathrm{Heaviside}(-\_U1\,v) + t\,\mathrm{Heaviside}(-vt))\Big)\Big)\,(2\,\mathrm{Heaviside}(M) - 1)\,\alpha\,\phi 0 \qquad \textbf{(18)}$$

$$L := 0.5 : M := 0.25 : \kappa := 130 : \rho := 2810 : cp := 960 : \alpha := \frac{\kappa}{\rho\cdot cp} : \phi 0 := 1500 :$$

For v = 50mm/min:

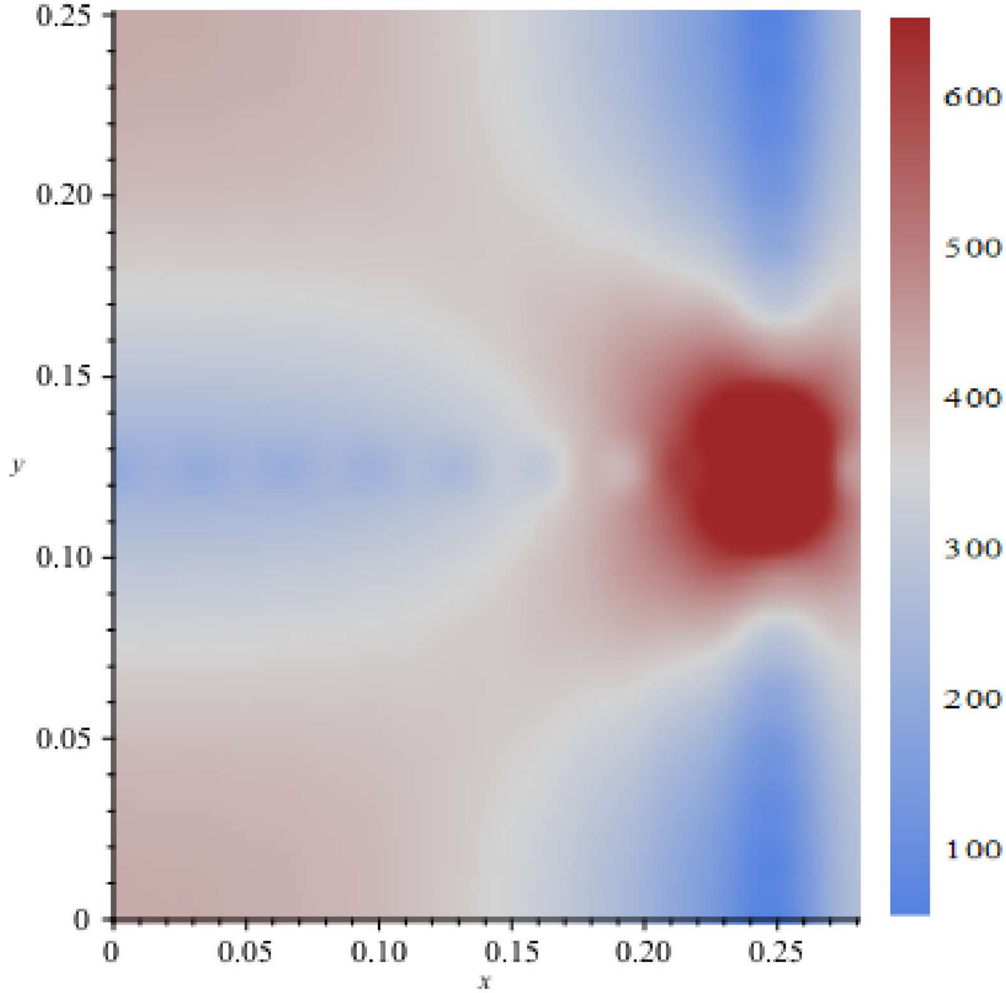
$$v := \frac{0.050}{60} : t := \frac{\mathrm{L}}{2\cdot v} :$$

$$u(x, y) := \frac{U0}{M\cdot L} + sum\left(sum\left(\frac{X[n]}{normsqX}\cdot\frac{Y[m]}{normsqY}\cdot U, n = 1..30\right), m = 1..30\right);$$

$$u := (x, y) \mapsto \frac{U0}{M\cdot L} + \left(\sum_{m=1}^{30}\sum_{n=1}^{30}\frac{X_n\cdot Y_m\cdot U}{normsqX\cdot normsqY}\right) \qquad \textbf{(19)}$$

with(*plots*) :

densityplot(*u*(*x*, *y*), *x* = 0 ..0.28, *y* = 0 ..*M*);



For v = 100 mm/min:

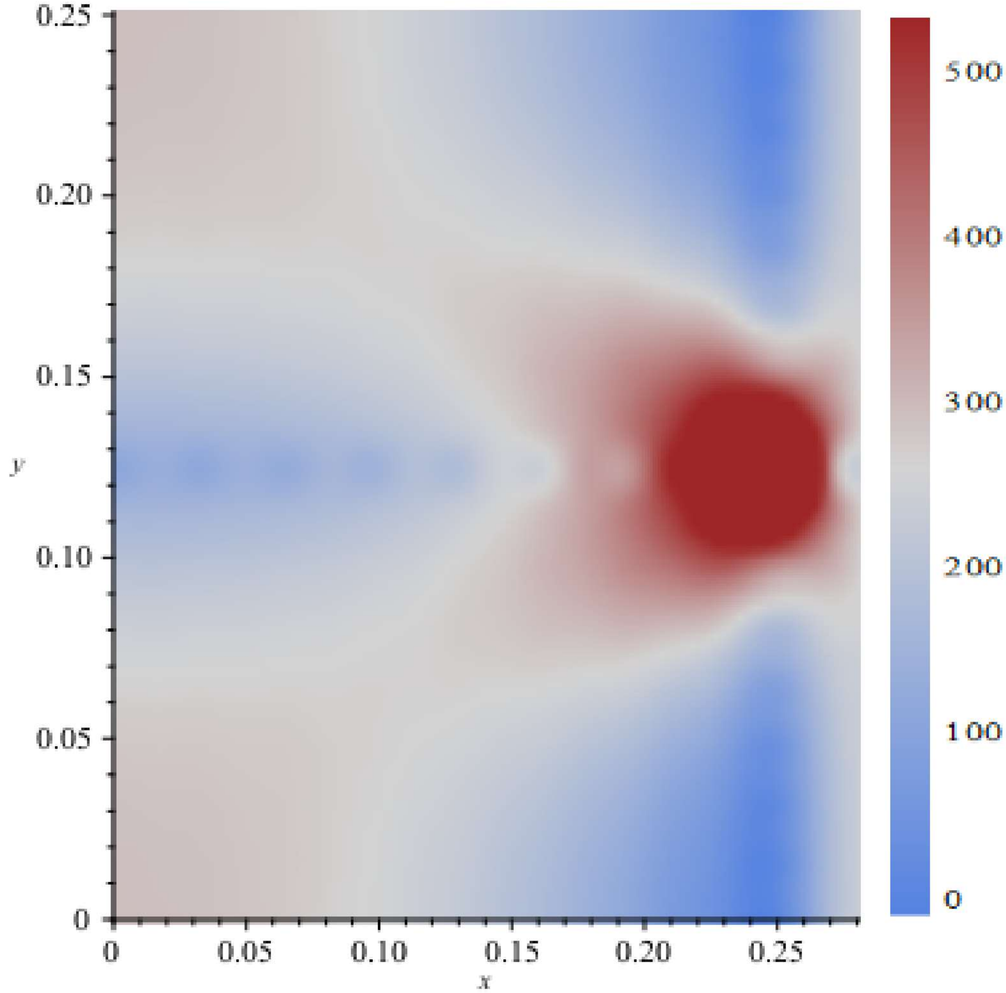
$v := \frac{0.100}{60} : t := \frac{\text{L}}{2 \cdot v} :$

$u(x, y) := \frac{U0}{M \cdot L} + sum\left(sum\left(\frac{X[n]}{normsqX} \cdot \frac{Y[m]}{normsqY} \cdot U, n = 1 .. 30\right), m = 1 .. 30\right);$

$$u := (x, y) \mapsto \frac{U0}{M \cdot L} + \left(\sum_{m=1}^{30} \sum_{n=1}^{30} \frac{X_n \cdot Y_m \cdot U}{normsqX \cdot normsqY}\right) \quad \textbf{(20)}$$

$with(plots) :$

$densityplot(u(x, y), x = 0 ..0.28, y = 0 ..M);$



For v = 300 mm/min:

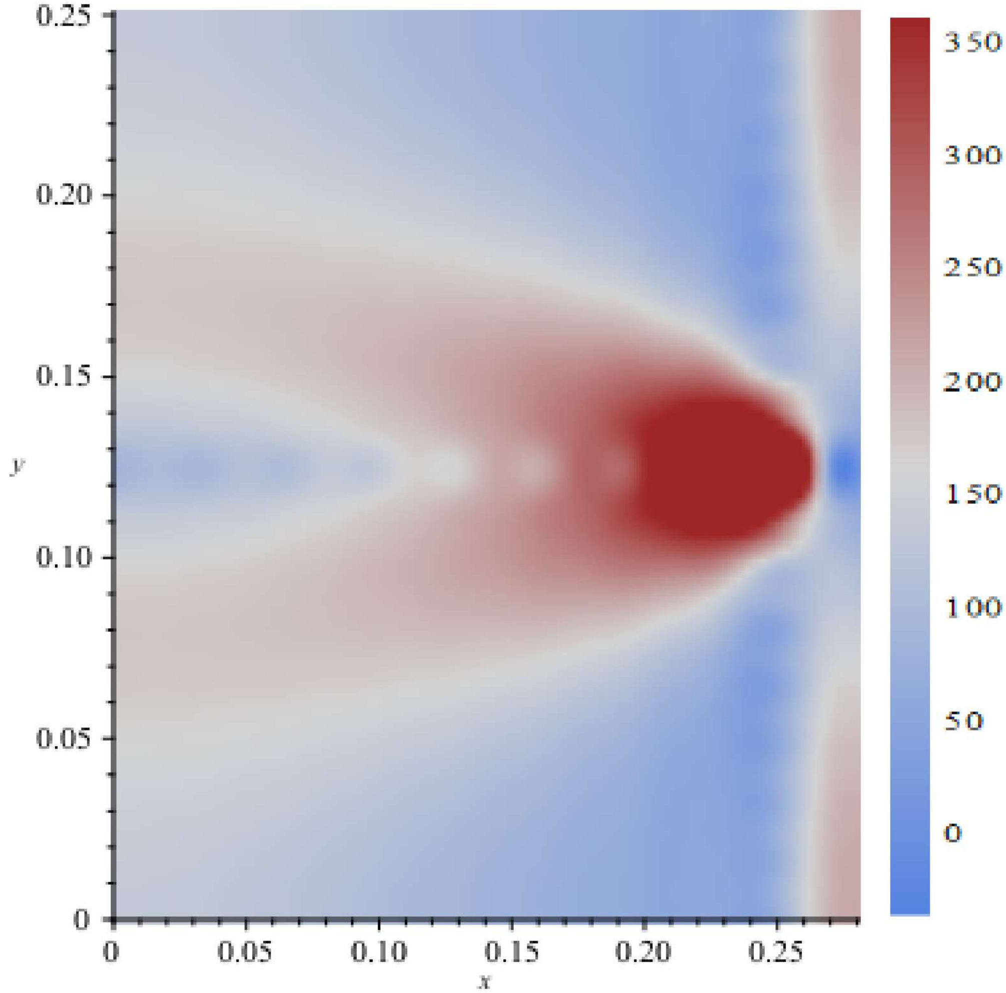
$v := \dfrac{0.300}{60} : t := \dfrac{\mathrm{L}}{2 \cdot v} :$

$u(x, y) := \dfrac{U0}{M \cdot L} + sum\left(sum\left(\dfrac{X[n]}{normsqX} \cdot \dfrac{Y[m]}{normsqY} \cdot U, n = 1 \,..\, 30\right), m = 1 \,..\, 30\right);$

$$u := (x, y) \mapsto \frac{U0}{M \cdot L} + \left(\sum_{m=1}^{30} \sum_{n=1}^{30} \frac{X_n \cdot Y_m \cdot U}{normsqX \cdot normsqY}\right) \quad \mathbf{(21)}$$

$with(plots) :$

$densityplot(u(x, y), x = 0 \,..0.28, y = 0 \,..M);$



For v = 500 mm/min:

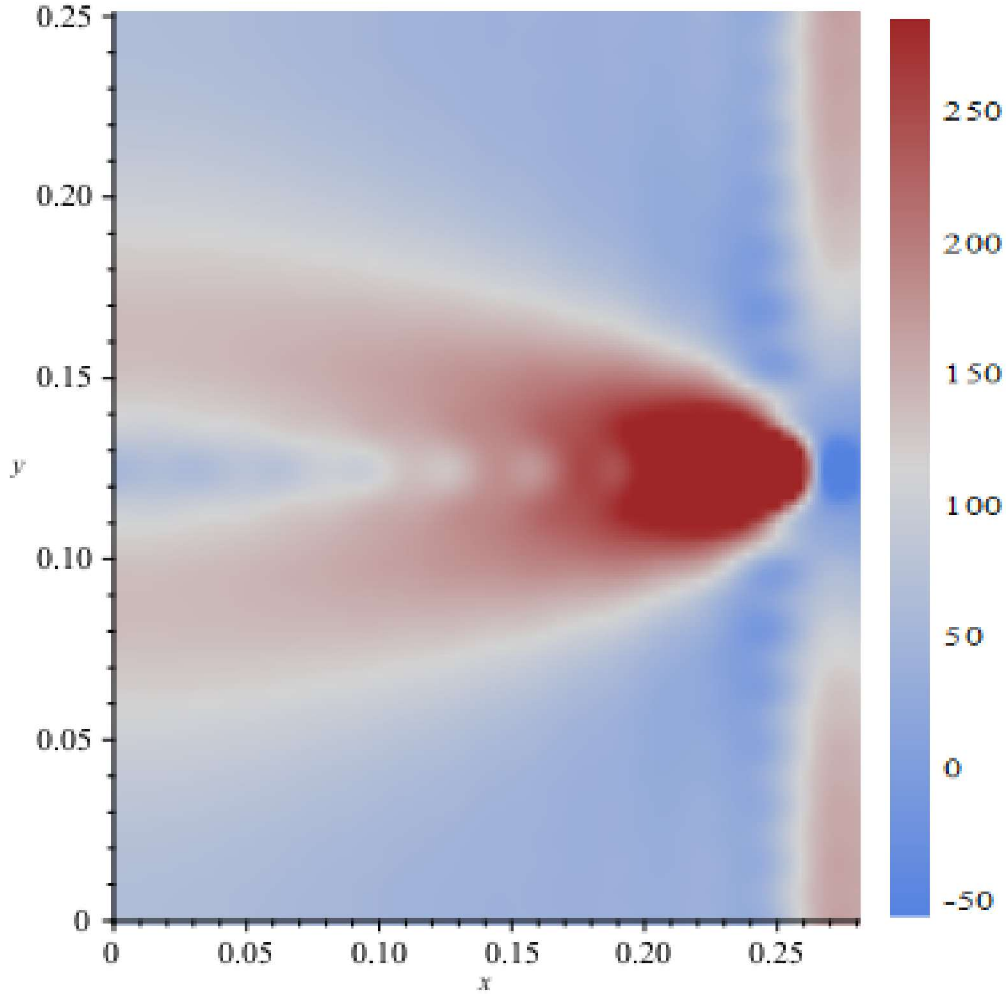
$v := \frac{0.500}{60} : t := \frac{\mathrm{L}}{2 \cdot v} :$

$u(x, y) := \frac{U0}{M \cdot L} + sum\left(sum\left(\frac{X[n]}{normsqX} \cdot \frac{Y[m]}{normsqY} \cdot U, n = 1 .. 30\right), m = 1 .. 30\right);$

$$u := (x, y) \mapsto \frac{U0}{M \cdot L} + \left(\sum_{m=1}^{30} \sum_{n=1}^{30} \frac{X_n \cdot Y_m \cdot U}{normsqX \cdot normsqY}\right) \tag{22}$$

$with(plots) :$

$densityplot(u(x, y), x = 0 ..0.28, y = 0 ..M);$



0.25
0.20
0.15
y
0.10
0.05
0
0
0.05
0.10
0.15
0.20
0.25
x
250
200
150
100
50
0
-50