Journal of Applied Engineering Mathematics
Volume 11, December 2024

Jet Turbine Nozzle Vibrations

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Abstract

Vibration often occurs in the nozzles of turbojets. In an effort to quantify the vibrations in a nozzle, this paper will solve second-order equation of motion of a thin cylinder using Sturm-Liouville Theory and a generalized Fourier series. A transient cylindrical model will be created from the solution.

Assumptions

Nozzle acts as a thin cylinder
No deflection in mid length supports
Axial symmetry
Negligible axial displacement

Nomenclature

θ	Angular coordinate
t	Time
R	Radius
x	Axial coordinate
h	Nozzle Length
s	Non-dimensional axial coordinate (x/h)
w	Radial displacement
u	Axial displacement
v	Circumferential displacement
ρ	Density
ν	Poison's ratio
E	Modulus of Elasticity

Introduction

The final component of a jet turbine is the nozzle. Since a nozzle converts the internal energy of the working fluid into thrust, nozzles deal with high temperatures and significant forces.

Nozzles are typically thinner than the other components of a jet turbine and can consist of dozens of parts to allow for mid-flight changes to their geometry. Both of these features make nozzles prone to fail by vibration. In fact, vibrations have the potential to jeopardize a nozzle far before forces would detach the nozzle from the nacelle. Thus, nozzles must be designed with minimal vibrations.

The purpose of this study is to quantify and plot the transient effects of vibration on a nozzle. This will be done through the second-order equation of motion for a cylindrical membrane. This is shown in equation (1).

$$\frac{\partial^2 w}{\partial s^2} = \frac{\rho(1-\nu)^2 R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{1}$$

The geometry and boundary conditions chosen will be based on the nozzle currently being designed by a BYU capstone team for a small-scale turbine.

Model

Formulation

The equation of motion for a thin cylinder are given in the coupled set of partial differential equations:[1]

$$\begin{cases} \frac{\partial^2 w}{\partial s^2} + \frac{(1-\nu)}{2}\frac{\partial^2 w}{\partial \theta^2} + \frac{(1+\nu)}{2}\frac{\partial^2 v}{\partial s \partial \theta} + \nu\frac{\partial u}{\partial s} \\ \qquad = \frac{\rho(1-\nu^2)R^2}{E}\frac{\partial^2 w}{\partial t^2} \\ \frac{(1+\nu)}{2}\frac{\partial^2 w}{\partial s \partial \theta} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial u}{\partial \theta} \\ \qquad = \frac{\rho(1-\nu^2)R^2}{E}\frac{\partial^2 v}{\partial t^2} \\ \nu\frac{\partial w}{\partial s} + \frac{\partial v}{\partial \theta} + u = \frac{\rho(1-\nu^2)R^2}{E}\frac{\partial^2 u}{\partial t^2} \end{cases} \tag{2}$$

Upon applying the assumptions, we can look at only the 1st equation in the couple, which reduces to a second-order PDE (1). Displacement, $w(s,t)$, can be treated as the product of a function of s and a function of t. The function of s and the function of t can be independently written in self-adjoint form and solved through the Sturm-Liouville solution pathway[2] given the boundary conditions below:

$$w(0,t) = 0 \tag{3}$$
$$w(1,t) = A\sin(\omega t) \tag{4}$$
$$w(s,0) = 0 \tag{5}$$
$$w(2/3,t) = 0 \tag{6}$$

Note that boundary conditions (3) and (5) are of the Dirichlet type and come from the nozzle being fixed to the engine and the nozzle starting with no displacement. Condition three is a function of natural frequency.[3] Condition (6) is the result of a rod that connects to the nozzle in order to actuate it. As stated earlier, we are assuming this rod does not deflect.

Solution

Recall that we are solving equation (1), which is a separable equation. We assume the solution is in the form

$$w(s,t) = X(s)T(t) \tag{7}$$

Plugging this into our original equation, we get

$$\frac{X''}{X} = \frac{\rho(1-\nu^2)R^2}{E}\frac{T''}{T} = \mu_n = -\lambda_n^2 \tag{8}$$

where μ is our separation constant. This creates two eigenvalue problems, one for X and one for T. Starting with the problem for X, we get

$$X'' + \lambda_n^2 X = 0 \tag{9}$$

By seeing that the characteristic equation has roots $\pm\lambda_n i$ we can determine that the solution for x is in the form

$$X_n(x) = c_1\cos(\lambda_n s) + c_2\sin(\lambda_n s) \tag{10}$$

Applying condition (3) to equation (10) can determine that $c_1 = 0$ so we are left with

$$X = c_2\sin(\lambda_n s) \tag{11}$$

We can then evaluate the eigenvalues by looking at condition (6).

$$X(2/3) = \sin(\frac{2\lambda_n}{3}) = 0$$

$$\lambda_n = \frac{3n\pi}{2}, n = 1, 2, ... \tag{12}$$

Now, we can look at the eigenvalue problem for T.

$$T'' + \frac{\lambda_n^2 E}{\rho(1-\nu^2)R^2}T = 0 \tag{13}$$

Using the characteristic equation, we determine the form of the solution is

$$T = c_3\cos(\lambda_n\sqrt{\frac{E}{\rho(1-\nu^2)R^2}}x) + c_4\sin(\lambda_n\sqrt{\frac{E}{\rho(1-\nu^2)R^2}}x) \tag{14}$$

We already know the eigenvalues, but we still need to determine the constants. Using condition (5) we determine that $c_3 = 0$, so our solution for T is now

$$T_n(t) = c_4 \sin(\lambda_n \sqrt{\frac{E}{\rho(1-\nu^2)R^2}} x) \quad (15)$$

Now, we can use equations (11) and (15) to form w as defined in (7).

$$w_n(s,t) = c_n \sin(\lambda_n s) * \sin(\lambda_n \sqrt{\frac{E}{\rho(1-\nu^2)R^2}} x) \quad (16)$$

combining c_2 and c_4 into c because they are both arbitrary constants. We can then define c using condition (4) and the Generalized Fourier Series. For time 0 to t_1, c is defined as

$$c_n = \frac{A \int_0^{t_1} \sin(\omega t) \sin(\lambda_n \sqrt{\frac{E}{\rho(1-\nu^2)R^2}} t)\, dt}{\sin(\lambda_n) \int_0^{t_1} \sin^2(\lambda_n \sqrt{\frac{E}{\rho(1-\nu^2)R^2}} t)\, dt} \quad (17)$$

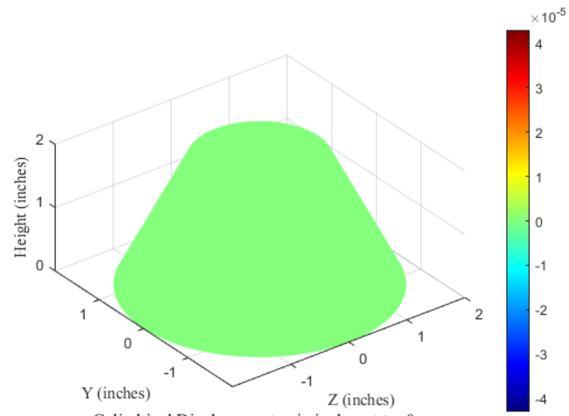
The final result is

$$w(s,t) = \sum_{n=1} w_n(s,t) \quad (18)$$

with c_n as defined in (17) and λ_n as defined in (12).

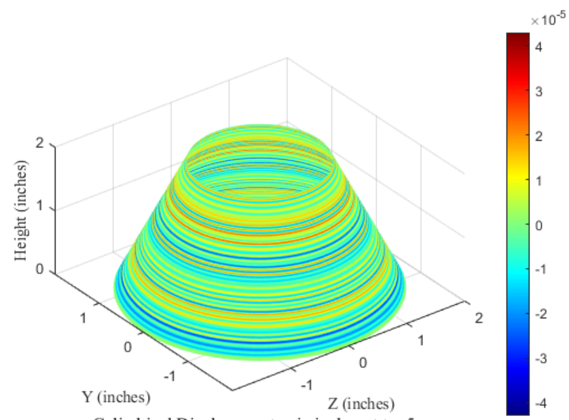
Results

The solution modeled in MATLAB using dimensions corresponding to a JETCAT P100RX turbojet nozzle made out of stainless steel. Fifty terms were used in the series for each value of w. Further information on the variables is contained in the comments of the code in the appendix.



Cylindrical Displacement w in inches at t = 0 s

Above is the nozzle at time $t = 0$ seconds. As we can see from the bar on the right, no displacements are present. Below is the nozzle at time $t = 5$ seconds.



Cylindrical Displacement w in inches at t = 5 s

Conclusions

Upon analyzing the displacements present in the graph, we conclude that they are on the order of micro-inches. Despite the presence of vibrations, the selected jet turbine nozzle geometry will be able to function with negligible vibrational effects.

References

[1] A. W. Leissa, “Vibration of Shells,” NASA, Washington D.C, 1973. Accessed: Dec. 04, 2024. [Online]. Available: https://www.vibrationdata.com/tutorials_alt/Leissa_vibration_shells.pdf

[2] Solovjov, Vladimir. “Integrated Engineering Mathematics.” ME505, Brigham Young University, 2021, https://www.et.byu.edu/ vps/ME505/IEM/08%2002.pdf

[3] Raj, K. Arul, et al. “Tribological and Vibrational Characteristics of Aisi 316L Tested at Elevated Temperature and 600torr Vacuum.” Defence Technology, China Ordnance Society, 18 June 2018, www.sciencedirect.com/science/article/pii/S2214914718300448.

Appendix

```matlab
clc;
clear;
clf;

% Parameters
A = 0.001;
h = 2;            % Height of the cylinder (in)
D = 1;               % Stiffness-like constant
rho = 1.189*10^-5;          % Density of air at 1472 F  (lbf/in^3)
nu = 0.25;           % Poisson's ratio for SST316L
E = 27.6 * 10^6;       % Young's modulus (psi)
n_terms = 50;        % Number of Fourier terms
t = linspace(0, 5, 200); % Time range (0 to 5 seconds)
x = linspace(0, h, 200); % Spatial domain (0 to h)
r = linspace(2, 1, 200); % Radial range
theta = linspace(0, 2 * pi, 100); % Angular range
omega = 120*2*pi; %natural angular frequency, see source 3

% Preallocate w
w = zeros(length(t), length(x));

% Generalized Fourier series calculation
for n = 1:n_terms
    lambda_n = (3 * n * pi) / 2; % lambda_n for nth term

    term_n = sqrt(E / (rho * (1 - nu^2) * r(n)^2));
    % Define a_n using numerical integration with improved precision
    numerator = A * integral(@(tau) sin(omega * tau) .*  ...
        sin(lambda_n * tau .* term_n), ...
        0, 5, 'RelTol', 1e-8, 'AbsTol', 1e-8); % High precision
    denominator = integral(@(tau) sin(lambda_n * tau .* term_n).^2, ...
        0, 5, 'RelTol', 1e-8, 'AbsTol', 1e-8); % High precision
    a_n = numerator / denominator;

    % Compute the nth term of w
    X_n = sin(lambda_n * x / h); % Spatial part
    T_n = a_n .* sin(lambda_n .* term_n .* t'); % Time-dependent part
    w = w + T_n * X_n; % Accumulate Fourier terms
end

% Cylindrical surface plot (r, theta, x)
[R, Theta] = meshgrid(r, theta);
X_cyl = R .* cos(Theta); % Cylindrical to Cartesian conversion
Y_cyl = R .* sin(Theta); % Cylindrical to Cartesian conversion
Z = repmat(x, length(theta), 1); % Height (x is along the height)

% Create a 3D matrix for W (time-dependent displacement)
W = zeros(size(X_cyl, 1), size(X_cyl, 2), length(t));
for i = 1:length(t)
    W(:, :, i) = repmat(w(i, :), size(X_cyl, 1), 1); % Match displacement
       to cylindrical coordinates
```

```matlab
end

% Plot the surface

for i = 1:length(t)
    surf(X_cyl, Y_cyl, Z, W(:, :, i), 'EdgeColor', 'none'); % Plot W on
       the surface
    colormap(jet);
    colorbar;
    caxis([-max(abs(w(:))) max(abs(w(:)))]);
    xlabel('Z (inches)','FontName', 'Times New Roman');
    ylabel('Y (inches)','FontName', 'Times New Roman');
    zlabel('Height (inches)','FontName', 'Times New Roman');
    text(0.5, -0.1, ['Cylindrical Displacement w in inches at t = ',
       num2str(t(i)), ' s'], ...
    'Units', 'normalized', ...                   % Normalized coordinates for
        positioning
    'HorizontalAlignment', 'center', ...         % Center alignment
    'FontName', 'Times New Roman', ...           % Times New Roman font
    'FontSize', 12);                             % Font size
    axis equal;
    pause(.001);
end
```

Listing 1: MATLAB Code for Cylindrical Displacement Analysis