Journal of Applied Engineering Mathematics

Volume 9, December 2024

BAKING LONG BREAD TRANSIENT THERMAL MODEL

Ammon Gallart **Caleb Swain**

Mechanical Engineering Department
Brigham Young University
Provo, UT 84602
ammon.gallart@gmail.com

caleb.c.student@gmail.com

ABSTRACT

Baking bread can be a rather problematic process since the bread can end up in a variety of shapes and sizes depending on the ingredients and method of baking. This process presents an intriguing heat transfer problem that we will be analyzing using theoretical models and 3D-graphing software.

INTRODUCTION

A cylindrical French bread begins as a small, short cylinder of raw dough which then expands and solidifies as it is heated. As it expands, the density lowers but the volume increases radially. We will be creating a simplified transient initial-value and boundary-value model of this problem to assess the validity of the theoretical method compared to raw data.

METHODS

Assumptions and Initial Data:

Given that baking introduces a plethora of variables, we will simplify this by assuming the following:

- The bread expands volumetrically from starting at 6 minutes until steady state occurs at 20 minutes. We assume a negative cosine curve.
- The surface area of the bread touching the baking pan is negligible compared to the surface area exposed to the heat of the oven
- The density of the bread is the same throughout the cylindrical volume
- This process follows the basic problem for a long cylinder (polar coordinates with angular symmetry)

Actual model:

The raw data comes from a study [1] that used thermocouples to measure the internal and external surface temperatures of the baking bread and compared them to the time spent baking.

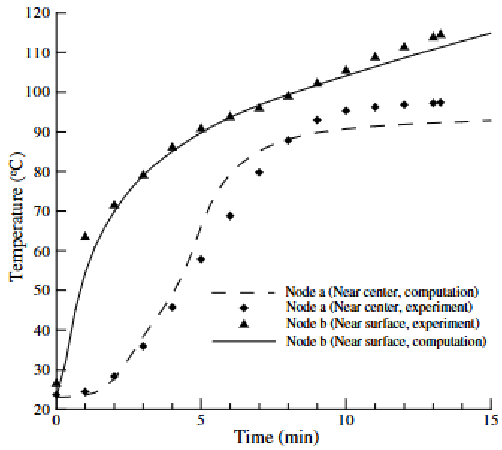

Figure 1: Temperature of bread over time

Theoretical model 1:

Using the basic case for the long solid cylinder with angular symmetry and correct assumptions and boundary conditions, we were able to model the baking bread as a function of radius and time.

Governing Equation:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}+F=\frac{1}{\alpha}\frac{du}{dt}$$

Assumptions:

- Long cylinder (French bread): du/dz = 0
- Radial symmetry (bread is on rack that is perfectly conductive): du/dθ = 0
- The heat source is not external nor internal, but is rather a boundary condition: F = 0
- α = 0.309 W/(m*K) = 0.00309 W/(cm*°C)
- u = R(r)T(t)

Boundary Conditions:

- $u(r=r_1(t), t>0)=U_{ext}=190℃$
- $u(r_i, t_1)=u(r_i, 0)=U_{dough}=27℃$

Initial Conditions:

- $t_1=0\ min, t_2=6\ min, t_3=20\ min$
- $r_i=1.825cm, r_f=2.175cm$

Analysis:

$$u=R(r)T(t)$$

$$\frac{R"}{R}+\frac{1}{r}\frac{R'}{R}=\mu=\frac{1}{\alpha}\frac{T'}{T}$$

$$r^2R"+rR'-r^2\mu R=0, \quad -\mu_n=\lambda_n^2, n=1,2\ldots$$

$$r^2R"+rR'+(\lambda^2r^2-0^2)R=0$$

$$R=c_1J_0(\lambda r)+c_2Y_0(\lambda r)$$

$$R_n(r)=c_1J_0(\lambda_n r), \quad \lim_{r=0} Y_0\to\infty, \quad c_2=0$$

$$R(r_i)=0=c_1J_0(\lambda_n r_i)$$

$$0=J_0(\lambda_n r_i)=J_0(1.825\lambda_n), \qquad c_1=1$$

$$R_n(r)=J_0(\lambda_n r)$$

$$\lambda_n=1.31771, 3.0247, 4{,}74177, 6.46111, 8.18132$$

$$p(r)=r, \quad weight\ function$$

$$T'-\alpha\mu T=T'+\alpha\lambda_n^2T=0$$

$$T_n(t)=c_3e^{-\alpha\lambda_n^2t}=e^{-\alpha\lambda_n^2t}, c_3=1$$

$$u(r,0)=u_0(r)=\sum_{n=1}^{\infty}a_nJ_0(\lambda_n r),$$

$$a_n=\frac{\int_0^{r1(t)}u_0(r)J_0(\lambda_n r)rdr}{\int_0^{r1(t)}J_0^2(\lambda_n r)rdr}$$

$$u(r,t)=\sum_{n=1}^{\infty}a_nR_nT_n=\sum_{n=1}^{\infty}a_nJ_0(\lambda_n r)e^{-\alpha\lambda_n^2t}$$

Since the first 4 eigenvalues are sufficient for calculating the approximate solution of u(r,t) for the baking long bread problem, only the first four eigenvalues will be used in the summation term for u(r,t).

First, we will see if just including the first eigenvalue will be sufficiently accurate in representing this heat transfer problem.

$$u(r,t)=\frac{\int_0^{r_1(t)}U_{ext}J_0(\lambda_1 r)rdr}{\int_0^{r_1(t)}J_0^2(\lambda_1 r)rdr}J_0(\lambda_1 r)e^{-\alpha\lambda_1^2t}$$

$$U_{ext}=190℃, \lambda_1=1.31771$$

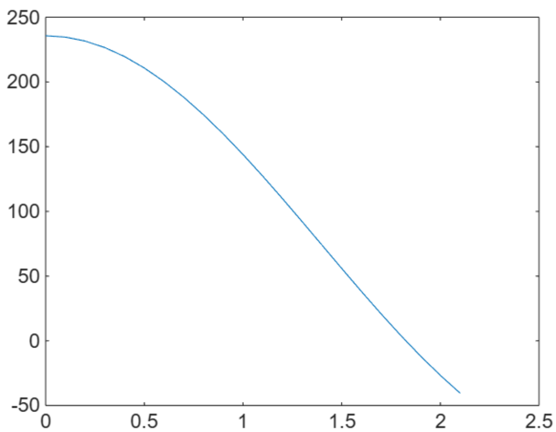

Figure 2: u(r,t) animation for the theoretical model 1. Vert.: Temp.(°C), Horiz.: Radius(cm)

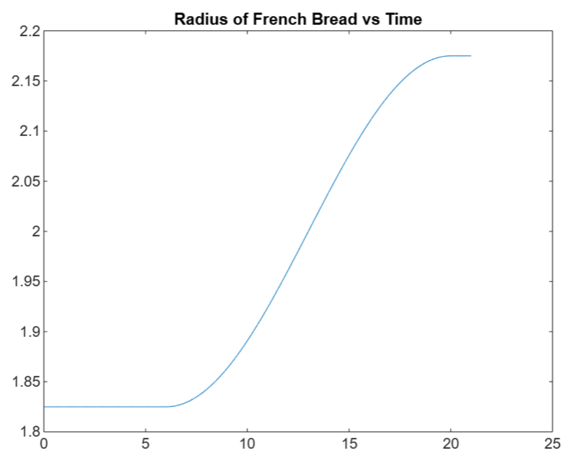

Figure 3: radius (cm, vertical axis) vs time (minutes, horizontal axis)

Theoretical model 2:

To be more accurate of the baking bread situation, we will now make the external surface temperature source a moving boundary condition, using comparative analysis from the Stefan Function Solution [2].

Governing Equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{du}{dr} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + F = \frac{1}{\alpha}\frac{du}{dt}$$

Assumptions:

- Long cylinder (French bread): du/dz = 0
- Radial symmetry (bread is on rack that is perfectly conductive): du/dθ = 0
- The heat source is not external nor internal, but is rather a boundary condition: F = 0
- α = 0.309 W/(m*K) = 0.00309 W/(cm*°C)
- u(r,t) is not able to have separation of variables since the radius depends on time

Boundary Conditions:

- $u(r = r_1(t), t > 0) = U_{ext} = 190℃$

Initial Conditions:

- $u(r_i, t_1) = u(r_i, 0) = U_{dough} = 27℃$

Knowns:

- $t_1 = 0\, min, t_2 = 6\, min, t_3 = 20\, min$
- $r_i = 1.825cm, r_f = 2.175cm$

Analysis:

Similarity variable, η, to handle time-dependent boundary $r_1(t)$:

$$\eta = \frac{r}{r_1(t)}, \eta \in [0,1], r \in [0, r_1(t)]$$

$$\frac{du}{dr} = \frac{du}{d\eta} * \frac{d\eta}{dr} = \frac{1}{r_1(t)} * \frac{du}{d\eta}$$

$$\frac{\partial^2 u}{dr^2} = \frac{\partial}{\partial r}\left(\frac{1}{r_1(t)} * \frac{\partial u}{\partial \eta}\right) = \frac{1}{(r_1(t))^2} * \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{du}{dt} = \frac{du}{d\eta} * \frac{d\eta}{dt} + \frac{\partial u}{\partial t}|_\eta$$

$$\frac{1}{(r_1(t))^2} * \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta r_1(t)} * \frac{du}{d\eta} = \frac{1}{\alpha}\left[\frac{\partial u}{\partial t}|_\eta - \frac{r\dot{r}_1(t)}{(r_1(t))^2} * \frac{du}{d\eta}\right]$$

Stefan Boundary Condition:

$$u(\eta = 1, t) = U_{ext}$$

$$\frac{\partial u}{\partial \eta}|_{\eta=0} = 0$$

Solution Form:

$$u(r,t) = g(t)h(\eta)$$

$$h''(\eta) + \frac{1}{\eta}h'(\eta) + \lambda^2 h(\eta) = 0$$

$$\frac{1}{g} * \frac{\partial g}{\partial t} + \frac{\dot{r}_1(t)}{r_1(t)}\eta\frac{h'(\eta)}{h(\eta)} = -\frac{\lambda^2}{\alpha}$$

General Solution:

- Spatial Component

$$h(\eta) = c_1 J_0(\lambda\eta) + c_2 Y_0(\lambda\eta)$$
$$c_2 = 0\ since\ Y_0\ diverges\ at\ \eta = 0$$
$$h(\eta) = c_1 J_0(\lambda\eta)$$
$$c_1 = 1\ since\ it\ cannot\ be\ non-zero$$
$$h_n(\eta) = J_0(\lambda_n\eta)$$

- Time Component

$$g(t) = e^{-\frac{\lambda^2}{\alpha}t}$$

- General Solution (combined), $\eta=r/r_1(t)$

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\lambda_n \frac{r}{r_1(t)}\right) e^{-\frac{\lambda_n^2}{\alpha}t}$$

Applying Boundary and Initial Conditions:

$$r = r_1(t), u(r_1(t), t) = U_{ext}$$

$$u(r_i, t_1) = u(r_i, 0) = f(r) = U_{dough}$$

$$A_n = \frac{\int_{r_i}^{r_f} f(r) J_0(\lambda_n \frac{r}{r_1(t)}) r dr}{\int_{r_i}^{r_f} (J_0(\lambda_n \frac{r}{r_1(t)}))^2 r dr}$$

DISCUSSION AND CONCLUSION

Given that theoretical model 1 predicted 262°C at 15 minutes, which was not like the empirical model from the cited paper. These theoretical models are approximate and may or may not include incorrect assumptions, boundary conditions, and/or initial conditions, so they may or may not be accurate in predicting the baking of long cylindrical bread in an oven. We learned that we cannot use boundary conditions to replicate the effects of a heat source. We will redo these calculations using a continuous heat source at the surface and that will move with the expanding surface.

Baking bread is not an exact science, since ovens and ingredients can vary widely between users. What is important is that the theoretical models contained the general trend that bread increases in size as it bakes over time. Future work can include more accurate assumptions, boundary conditions, and initial conditions.

REFERENCES

[1] J. Zhang and A. K. Datta, "Mathematical modeling of bread baking process," Journal of Food Engineering, vol. 75, no. 1, pp. 78–89, Jul. 2006, doi: https://doi.org/10.1016/j.jfoodeng.2005.03.058.

[2] Wikipedia Contributors, "Stefan problem," *Wikipedia*, May 13, 2024.

Appendix A: MatLab Code for Model 1

```
t_i = 6; % initial time,
sec
t_f = 20; % final time, sec
t = t_i:0.1:t_f; % time
vector

alpha = 0.00309;

r_i = 3.65/2; % cm
r_f = 4.35/2; % cm
r_avg = (r_i+r_f)/2; %
average radius
r_mag = (r_f-r_i)/2; %
amplitude of cosine curve
r_1(1:61) = r_i;
r_1(61:201) = -
r_mag*cos(pi/14*(t-
6))+r_avg;
r_1(201:211) = r_f;

t = 0:.1:21;
figure(1)
plot(t,r_1)
title('Radius of French
Bread vs Time')

for i = 1:211 % index for
time
    % t(i) = t(i);
    % r_1(i) = r_1(i);
    r = 0:0.1:r_1(i);
    u = zeros(size(r)); %
initialize u
    % for n = 1:4 % index
for eigenvalues
        lambda = 1.31771; %
solve for lambda
        a_top = @(r)
190*besselj(0,lambda*r).*r;
        a_bottom = @(r)
besselj(0,lambda*r).^2.*r;
        a =
integral(a_top,0,r_1(i))/in
tegral(a_bottom,0,r_1(i));
        R =
besselj(0,lambda*r);
        T = exp(-
alpha*lambda^2*t(i));
        u = a*R*T + u;
    % end
    plot(r,u)
    pause(0.1)
end
```

