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Exploring Development of Shock Waves in Fluids Using the Inviscid Burger's PDE

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Abstract

A shockwave is a region of sharp discontinuities between properties within a fluid. To model formation of a shockwave from an initial condition, a simple PDE will be developed using Navier-Stokes and the method of characteristics. Two starting conditions, one linear and the other a Gaussian distribution, were investigated to determine velocities after several seconds. It was found that with simple initial conditions, the velocity profile can be solved analytically, but yields unrealistic results. The reverse is true for higher order initial conditions, in which more realistic flow properties occur but need to be solved numerically.

Nomenclature

- ν - Kinematic Viscosity
- ρ - Density of fluid
- u - Velocity profile in horizontal (x) direction
- v - Velocity profile in vertical (y) direction
- w - Velocity profile in z direction
- t - Time in seconds.
- x - Distance along horizontal direction
- y - Distance along vertical direction
- z - Distance along z direction
- M - Mach number, or velocity divided by speed of sound of medium

Introduction

Fluids like air and water are often thought of as continuous, free flowing substances that gradually change their properties from one location to another, and over time. Indeed, this is usually the case for most scenarios. However, when extreme pressures, velocities, and temperatures are involved, these gradual changes can become very abrupt discontinuities known as shock waves. This can often happen when an object is traveling at sonic speeds. In short, because the object is traveling faster than the speed of sound, the fluid cannot transmit properties from one point to another ahead of the object. Thus, a barrier-like wave can form in front of the object, where the properties will "jump" as soon as one crosses the wave. Mathematically these are infinite jumps, although in reality the jumps occur over a short span. A visual example of a shock wave, or bow shock, can be seen below. A reentry capsule, simulated in Star CCM, a computational fluid dynamics program, traveling at Mach 12 produces a "bow shock" in front if it. This paper will investigate the velocity profile over the time span it takes for the shockwave to develop.

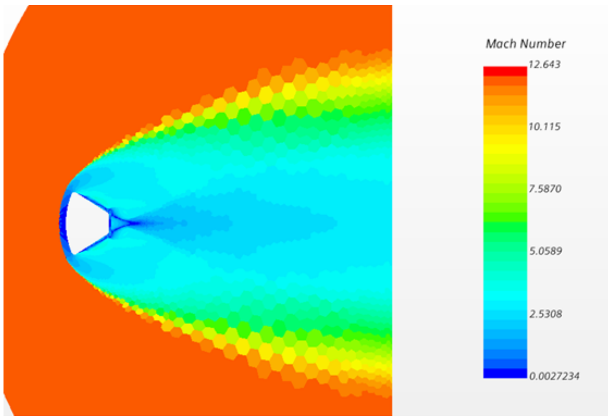

Figure 1: A supersonic reentry capsule travelling through the atmosphere at Mach 12, simulated in CFD.

Methods and Results

We will start developing our PDE by looking at the Navier-Stokes momentum equation in the x-direction, given by the following equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

We will now make some assumptions. For each scenario, we will assume 1-D flow, canceling out all terms involving v, w, y and z. For simplicity, we will ignore viscosity, although in real flow scenarios viscosity can have an effect on shockwave properties. Most notably, viscosity makes the gradient across the shockwave more gradual. Finally, we will assume there is no pressure gradient (dp/dx) and ignore changes in density, as each scenario will involve modest velocities. We are now left with a simple PDE below. Another name for this is the Inviscid Burger's Equation. The standard Burger's equation contains a viscosity term with a double derivative, but it is much more difficult to solve. It requires transforming the PDE into an ODE using the Cole-Hopf Transformation, and then solving the ODE with the standard approach for a wave equation. However, the reverse transformation from the Cole-Hopf Transformation can be cumbersome to solve analytically.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$$

Note that this equation is a PDE as the equation is a function of both position and time. It is actually possible to obtain a general solution for position dependent initial conditions using the approach from this paper [2].

We will now proceed to solve for u(x,t) using the method of characteristics [3]. We need an initial condition for the velocity profile at t = 0. For now we will denote it as F(x), as follows:

$$u(x,0)=F(x)$$

We will now use the method of characteristics to reduce this PDE into an ODE. We need to find the curve such that:

$$\frac{du(x(t),t)}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$$

By performing the multi variable chain rule we get

$$\frac{du(x(t),t)}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}$$

By comparing the two equations, we can deduce that u = dx/dt. This is known as a characteristic equation. We now see that du/dt must also equal 0. This is the second characteristic equation. Integrating that, we get:

$$x=ut+C$$

We can find C by using our initial condition u(x,0),leading to

$$C=x_0\rightarrow x=ut+x_0,\ or\ x_0=x-ut$$

We can now see that because because u depends on our initial condition F(x), we can now derive the general solution for the inviscid Burger's Equation:

$$u(x,t)=F(x-ut),\ where\ u(x,0)=F(x).$$

The equation is implicit, meaning it can be solved analytically for some (but few) cases, but must be solved numerically for others. Let us first look at a case that can be solved analytically. The simplest possible equation is when F(x) = x. Linear equations like this allow isolation of u easily, giving a solution of:

u(x,0) = x/(1 - t)

A plot of u(x,t) at various timesteps is shown in the following figure:

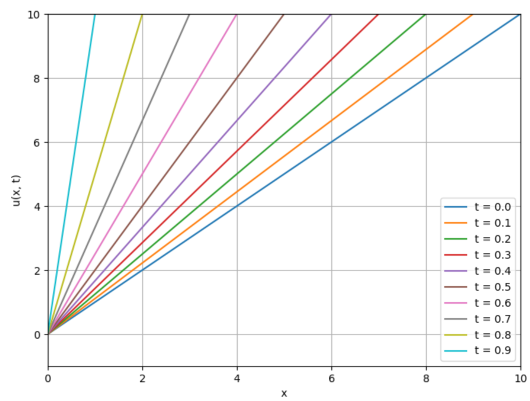

Figure 2: u(x,t) plot for an initial condition u(x,0) = F(x) = x.

While this chart is not very exciting, it shows that after a certain time t (1 second) in this case, the velocity gradient becomes infinitely steep, which is technically a shockwave as an infinite discontinuity occurs. Limiting the solution to a purely linear one is nonphysical and does not give much insight otherwise. Unfortunately, not many equations more complex than linear ones can be solved analytically. However, the PDE can easily be approximated with a numerical solution for many complex initial conditions. As a more realistic example, a popular initial condition that more accurately shows shock formation is the initial condition, or the Gaussian curve:

$u(x, 0) = e^{-\frac{x^2}{2}}$

Using a simple algorithm in python that uses discrete time steps, we can derive the following plot:

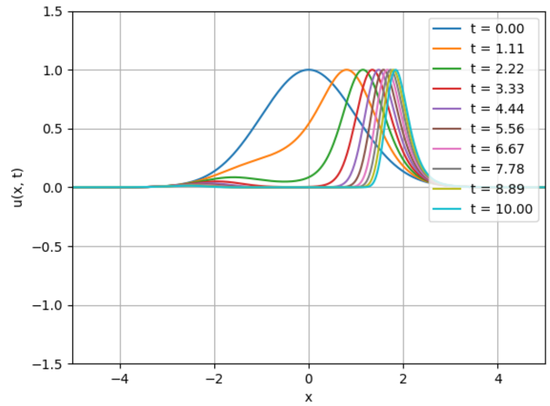

Figure 3: u(x,t) plot for an initial condition F(x) = $e^{-\frac{x^2}{2}}$.

From the figure we see that the curve becomes much more compressed towards the right, and the velocity gradient begins to steepen. It seems that mathematically, a shockwave takes an infinite time to develop as it approaches an asymptotic solution. Although this is not fully realistic, it shows how the fluid flow field adjusts itself to satisfy the momentum equations.

Conclusions

Solving even the simplest PDE's can be a daunting task. However, they are extremely useful for modelling unsteady problems including shock formation. Shock waves are remarkably complicated phenomena, and it is hard to model them exactly unless advanced CFD software is used. However, simple PDE's like Burger's Equation can provide a general visualization of a fluid's ability to adjust to an extreme force acting on it. From our analysis, it is clear that more advanced initial conditions are required to generate realistic plots. However, the complexities of these initial conditions usually mean the equation remains implicit and must be solved analytically.

Acknowledgements

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References

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2 —. Inviscid Burgers' Equation. 2016.

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