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DYNAMIC HEAT PROPAGATION IN A METAL PAN: INSIGHTS FROM A ONE-DIMENSIONAL BESSEL FUNCTION MODEL

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ABSTRACT

This study presents a mathematical model for the heating process of a metal pan placed on a gas burner, utilizing the one-dimensional heat equation in polar coordinates. The model combines transient and steady-state solutions, employing Fourier's law and separation of variables, with boundary conditions accounting for the heat input at the pan's base and convective heat loss to the surroundings. The results reveal the dynamic temperature distribution within the pan over time, providing critical insights into optimizing pan design and materials for enhanced thermal performance. This research bridges material science and culinary technology, advancing thermal management in practical applications.

Figure.1: A frying pan on a gas stove (*onceuponachef.com*)

NOMENCLATURE

- *T(r,t):* Temperature at a distance r from the center at time t .
- : Initial temperature of the pan
- : Base temperature of the pan at the heat source.
- : Steady-State temperature.
- : Transient temperature.
- : Radial distance from the center of the pan.
- : Radius of the pan.
- : Time
- k: Thermal conductivity (W/m·K)
- α: Thermal diffusivity (m^2/s)
- h: Convective heat transfer coefficient (W/m^2·s)

INTRODUCTION

Figure.2: A frying pan on a gas stove (*unclebuffalo.com*)

Heating a metal pan on a gas burner is a routine activity, whether in cooking or industrial processes. Although seemingly simple, understanding how heat travels through the pan is essential for optimizing cookware design and energy efficiency. When heat is applied at the base, it spreads through the material by conduction, while the top surface loses heat to the surrounding air via convection. This interplay of heat transfer mechanisms is critical for achieving uniform temperature distribution and minimizing energy loss.

This study develops a mathematical model to analyze the heating process using the one-dimensional heat equation in polar coordinates. By combining transient and steady-state solutions with boundary conditions reflecting heat input and convective losses, the model provides a detailed temperature profile over time and space. Employing Fourier's law and separation of variables, the work simplifies complex thermal dynamics into an accessible framework.

The findings address practical challenges in designing cookware with enhanced thermal properties and contribute to material science by identifying key parameters affecting heat transfer efficiency. The results not only guide the development of better-performing pans but also have broader applications in culinary technology and energy-efficient heat management systems. By bridging theoretical modeling and practical considerations, this research lays the foundation for advancements in cookware design and other heat-transfer applications.

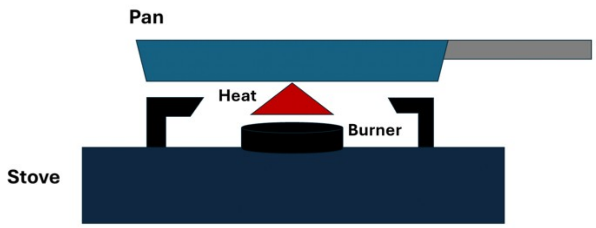

Figure.3: Simple diagram of common household gas burner.

Mathematical Model

The heat transfer in the pan is governed by the heat equation, One - Dimensional Heat Equation in Polar Coordinates:

Where *T (r, t)* is the temperature at radial distance *r* and time *t.* is the thermal diffusivity of the pan material, *k* is the thermal conductivity, c_p is the specific heat capacity and is the density.

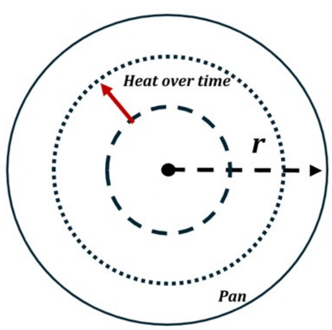

Figure.3: Propagation of heat throughout the pan with respect to time.

To solve for this equation boundary and initial conditions are given as follows:

Boundary Condition:

- At the center of the pan (*r* = 0), the temperature is constant due to the heat source:

- At the edge of the pan (*r* = *L)*, the temperature is fixed at the heat source:

Initial Condition:

- At *t* = 0, the temperature throughout the pan is uniform:

Using the method of separation of variables, it is assumed the solution is the sum of a steady-state solution $T_{ss}(r)$ and transient solution $T_t(r,t)$:

Steady-State Solution

At steady state the equation simplifies to:

The general solution is:

Applying boundary conditions:

- At r = 0, T_{ss} (r) must remain finite. Hence, C_1=0.

- At r = L, T_{ss} (L)=T_b Therefore, $C_2 = T_b$

Thus, the *steady-state solution* is:

Transient Solution:

Subtract the steady state solution to *T (r, t)* :

Using separation of variables let:

Divide through :

This separates into two ordinary differential equations:

1. Time – dependent equation :

Since the temperature will be increasing over time G(t)1. This term starts at 0 when t = 0 (no heating has occurred yet) and asymptotically approaches 1 as t→∞.
Thus:

Radial – dependent equation:

Radial Solution

The radial equation is a Bessel differential equation. Its general solution is:

where J_0 is the zeroth-order Bessel function of the first kind.

The boundary condition at r =L gives:

where λ_n are the roots of J_0.

Using the Fourier Series can simplify the equation from the use of separation of variables in determining the transient temperature profile. It can decompose the initial condition into sets of simpler components that can be individually analyzed and then recombined to find the overall solution to the heat equation over time.

In relation to this we can change the terms into:

In result the transient solution profile would be:

Solving for C_n:

Where J_1 is a first-order Bessel Function of the first kind. Here, the denominator shows because is related to the orthogonality property of the Bessel functions.

The complete the solution for the temperature profile it is given that:

Combining and using the solutions solved for steady-state and transient profiles, the final temperature profile would be:

The final temperature profile describes how heat propagates radially outward from the center of the pan over time, starting at a high temperature near the heat source and decreasing toward the edges. The profile is determined by solving the one-dimensional heat equation in polar coordinates, using a series expansion involving Bessel functions to account for the geometry of the pan and the thermal diffusivity of the material. This model provides a comprehensive understanding of heat transfer dynamics, emphasizing the influence of thermal properties and boundary conditions.

CONCLUSIONS

This study successfully modeled the radial heat transfer in a metal pan heated by a gas burner using the one-dimensional heat equation in polar coordinates. The resulting temperature profile, derived as a series solution involving Bessel functions, accurately captured both the transient and steady-state behavior of heat propagation. The analysis highlighted how thermal diffusivity, heat source intensity, and boundary conditions shape the spatial and temporal evolution of temperature. The model offers practical insights into optimizing cookware design for more uniform heat distribution, improved energy efficiency, and enhanced performance. Future extensions of this work could explore multi-dimensional geometries, variable thermal properties, and more complex boundary conditions, enabling broader applications in material science, energy systems, and thermal engineering.

ACKNOWLEDGMENTS

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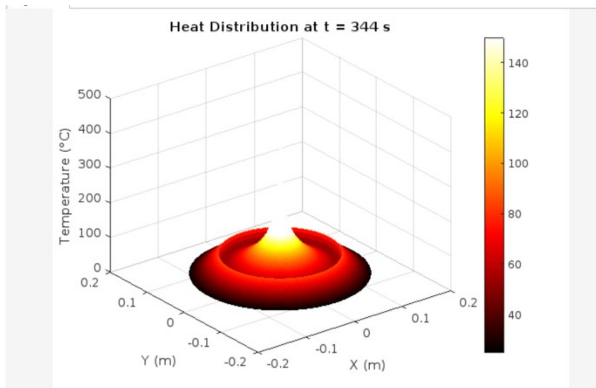
APPENDIX

Matlab Animation Code




```
% Parameters
L = 0.15; % Radius of the pan (meters)
T_initial = 25; % Initial temperature (°C)
T_burner = 150; % Temperature at the heat source (°C)
alpha = 1.1e-5; % Thermal diffusivity (m^2/s)
time_steps = 600; % Total time in seconds
dt = 1; % Time step (seconds)
n_terms = 50; % Number of terms in the series expansion
r = linspace(0, L, 100); % Radial domain (m)
t_values = 0:dt:time_steps; % Time domain (s)

% Compute the first n_terms roots of J0 manually
lambda_n = zeros(n_terms, 1); % Pre-allocate
syms x;
for n = 1:n_terms
    if n == 1
        lambda_n(n) = fzero(@(x) besselj(0, x), [0, 5]); % Find the first root
    else
        % Use previous roots to bracket the next root
        lambda_n(n) = fzero(@(x) besselj(0, x), [lambda_n(n-1) + 0.1, lambda_n(n-1) + 5]);
    end
end
lambda_n = lambda_n / L; % Scale roots for the given radius

% Compute the temperature profile using the series solution
T_series = zeros(length(r), length(t_values)); % Pre-allocate temperature matrix

for t_idx = 1:length(t_values)
    t = t_values(t_idx); % Current time
    for i = 1:length(r)
        r_val = r(i); % Current radial position
        T_sum = 0; % Initialize the series sum
        for n = 1:n_terms
            % Compute Bn and the transient term
            Bn = (2 * (T_burner - T_initial)) / (lambda_n(n) * besselj(1, lambda_n(n)))^2;
            T_sum = T_sum + Bn * besselj(0, lambda_n(n) * r_val) * (1 - exp(-alpha * lambda_n(n)^2 * t));
        end
        T_series(i, t_idx) = T_initial + T_sum; % Add initial temperature
    end
end

% 3D Animation for Surface Heat Distribution
figure;
theta = linspace(0, 2*pi, 100); % Angular domain
[R, TH] = meshgrid(r, theta); % Convert radial coordinates to polar
[X, Y] = pol2cart(TH, R); % Convert polar to Cartesian for 3D plotting

for t_idx = 1:length(t_values)
    Z = repmat(T_series(:, t_idx)', size(X, 1), 1); % Heat distribution matrix

    % Plot the surface
    surf(X, Y, Z, 'EdgeColor', 'none');
    colormap hot;
    colorbar;
    caxis([T_initial T_burner]);
    xlabel('X (m)');
    ylabel('Y (m)');
    zlabel('Temperature (°C)');
    title(['Heat Distribution at t = ', num2str(t_values(t_idx)), ' s']);
    view(3); % 3D view
    pause(0.05); % Pause for animation effect
end

disp('3D Animation completed.');
```