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Analytical Solution for the Euler-Bernoulli Beam Theory with Damping for Two Identical Bolted Beams

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Abstract

This paper presents an analytical approach to solving the Euler-Bernoulli beam equation with linear damping, applied to a system of two identical beams bolted together. Bolted systems are widely used in structural applications due to their modularity, but their dynamic behavior, especially energy dissipation through damping, requires further investigation. By employing the integral transform method and leveraging solutions strategies from foundational PDE resources, this study derives eigenfunctions and eigenvalues to construct a solution for the damped beam equation under specific boundary and initial conditions. Results are generated for an impulse impact force on the system.

Nomenclature

μ	eigenvalues of supplementary ODE
ω_d	damped natural frequencies
ρ	linear density
c	damping coefficient
E	young's modulus
h	height of beam
I	second moment of area
L	length of beam
$S(x,t)$	source impulse
$u(x,t)$	deflection of beam
w	width of beam
x_p	location of point impulse

Introduction

The Euler-Bernoulli beam theory is the cornerstone of structural mechanics, providing a framework to describe bending and deformation in slender beams. A wide variety of structural problems can be simplified to a beam-like model, enabling solutions in both static and dynamic contexts. Moreover, the theory can be extended to include damping effects, which significantly influence a beam's dynamics response and energy dissipation characteristics.

This paper focuses on solving the Euler-Bernoulli beam equation with damping analytically for a system of two identical beams bolted together. Bolted systems are widely used for modularity and adaptability; however, bolted connections introduce unique dynamic characteristics that warrant a detailed study. Damping, in particular, plays a significant role in these systems as energy is lost in the slippage of the joint. Although this paper will not address the slippage of the joint, it will provide a good baseline for comparison and parameter relationship for the system.

In developing the analytical solution, this study uses the solution strategies found in [3] to solve partial differential equations and builds upon previous work, such as Black and Blackham's investigation using the integral transform method [1]. While their work demonstrates the effectiveness of using the integral transform, this work will extend it to include damping.

The object of this paper is first to derive an analytical solution for the Euler-Bernoulli beam equation with damping. Second, it will apply the solution to a bolted beam system and supply some results due to an impulse input force.

The remainder of this paper is organized as follows. Sections 3-6 outlines the methodology, including the governing equations, boundary conditions, and analytical solution process. Section 7 presents the results and their discussion. Finally, Section 8 offers concluding remarks.

Problem Statement

The system under consideration consists of two identical beams that are bolted together on both ends. The beams are assumed to be identical so a symmetry condition is assumed to simplify the problem to solve for only one beam at a time. Figure 1 illustrates the physical setup of the bolted system.

The Euler-Bernouli beam equation governing the displacement of the beam [2] with an added linear damping term is given by:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = -\rho\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} + S$$

where E, I, ρ and c are young's modulus, second moment of area, linear density, and damping ratio respectively. u is the displacement and is a function of length and time, $u(x,t)$, $c\frac{\partial u}{\partial t}$ is the added linear damping term, and S is a source term function of length and time, $S(x,t)$.

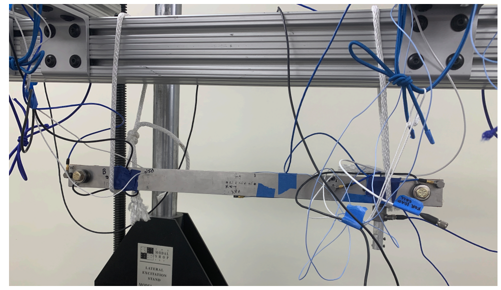

Figure 1: Two beams bolted together at each hanging from a test rig.

The beams have a constant cross-section, so the equation can be simplified further by pulling the flextural rigidity terms out of the partial.

$$EI\frac{\partial^4 u}{\partial x^4} = -\rho\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} + S \tag{1}$$

The beam is initial at rest, so the following two initial conditions will be assumed to account for the second partial in time.

$$u(x,0) = 0 \tag{2a}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \tag{2b}$$

Four boundary conditions are defined to account for the fourth partial of x. The symmetry of the bolted beam system allows us to assume no displacement at the ends. The bolts clamp the two beams together at the ends, so the slope, which corresponds to the derivative of deflection, can be assumed to be zero at the ends. The boundary conditions can be summarized as follows:

$$u(0,t) = 0 \tag{3a}$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \tag{3b}$$

$$u(L,t) = 0 \tag{3c}$$

$$\frac{\partial u(L,t)}{\partial x} = 0 \tag{3d}$$

The load conditions will be approximate as an impulse tap on the structure to excite vibration and will be modeled as

$$S(x,t) = \delta(x - x_p)\delta(t)$$

where x_p is the point of the impulse force.

The following parameters will be used in this study

$$\begin{aligned} \rho &= 8000 & E &= 200 \times 10^{-9} \\ h &= 0.025 & w &= 0.003 \\ c &= 1000 & L &= 0.3 \\ x_p &= 0.4L \end{aligned}$$

where h and w is the height and width of the beam respectively. Then the area moment can be calculated as

$$I = h * w^3/12$$

Eigenvalue Problem

Due to the added source function, the PDE will not be solvable utilizing separation of variables. A generalized finite integral transform will be constructed of eigenfunctions with respect to the space term 'x'. The supplementary eigenvalue problem with its corresponding boundary conditions is

$$\frac{d^4X}{dx^4} = \mu X \tag{4a}$$
$$X(0) = 0 \tag{4b}$$
$$\frac{\partial X(0)}{\partial x} = 0 \tag{4c}$$
$$X(L) = 0 \tag{4d}$$
$$\frac{\partial X(L)}{\partial x} = 0 \tag{4e}$$

The differential equation is a linear constant-coefficient ODE, therefore, assuming $X = e^{\lambda x}$ give the following characteristic equation

$$\lambda^4 - \mu = 0$$

Solving for the roots of λ give

$$\lambda = \pm\sqrt[4]{\mu}, \pm i\sqrt[4]{\mu}$$

The general solution can then be written as

$$\begin{aligned} X(x) =& c_1 \sin\lambda x + c_2 \cos\lambda x \\ &+ c_3 \sinh\lambda x + c_4 \cosh\lambda x \end{aligned}$$

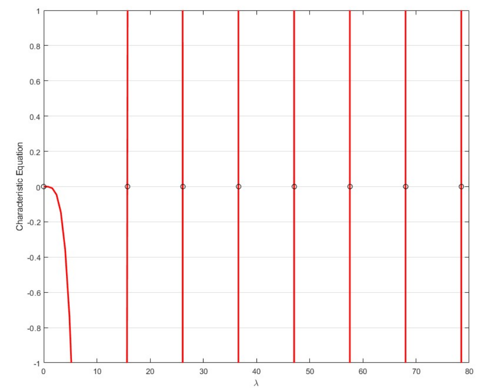

Figure 2: Plot of characteristic equation of the first six eigenvalues, λ_n

Applying boundary conditions 4b and 4c gives the relationships

$$c_2 = -c_4$$
$$c_1 = -c_3$$

Then after applying the boundary condition 4d the general solution can be simplified with by defining

$$\beta = -\frac{c_3}{c_4} = \frac{\cosh\lambda L - \cosh\lambda L}{\sinh\lambda L - \sin\lambda L} \tag{5}$$

Now when applying the last boundary condition 4e, $c_4 \neq 0$ otherwise it will result in a trivial solution, therefore the eigenvalues, λ, must be the roots of the following characteristic equation

$$-1 + \cos\lambda L = 0 \tag{6}$$

This function is plotted in Figure 2. The function has an infinitely many roots so functions involving the eigenvalues, λ_n will now be denoted with the subscript 'n', where $n = 1, 2, 3, \ldots$

The eigenfunctions can now be expressed as

$$\begin{aligned} X_n(x) =& -c_4[\cos\lambda_n x + \cosh\lambda_n x \\ &- \beta_n(\sinh\lambda_n x - \sin\lambda_n x)] \end{aligned} \tag{7}$$

The first four eigenfucntions can be seen in Figure 3, and are the normal mode shapes of the beam [2].

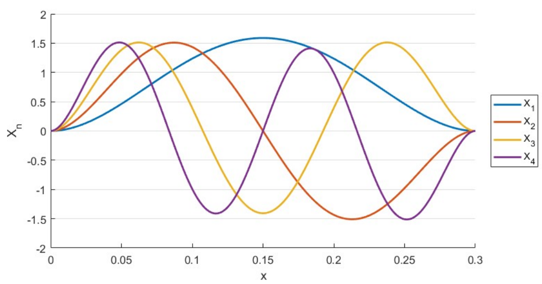

Figure 3: Plot of the first four eigenfunction, $X_n(x)$

Construct Finite Integral Transforms

A Hilbert Space $L^2(0, L)$ was then defined with the inner product given by

$$(v, w)_p = \int_0^L v(x)w(x)p(x)$$

and the norm

$$||v||^2 = (v, v)_p = \int_0^L v^2\, dx$$

where $p(x) = 1$ is the weight function.

A generalized Fourier series is then formed in terms of the eigenfunction from Equation 7 as

$$v(x) = \sum_{n=0}^{\infty} \frac{(v, X_n)_p}{||X_n||^2} X_n$$

The finite integral transform and its inverse are now constructed as

$$\mathcal{J}\{v(x)\} = (v, X_n) = \int_0^L v(x)X_n(x)\, dx = \bar{v}_n \quad (8)$$

$$\mathcal{J}^{-1}\{\bar{v}_n\} = \sum_{n=0}^{\infty} \frac{\bar{v}_n X_n(x)}{||X_n||^2} = v(x) \quad (9)$$

The operational property is then defined as

$$\mathcal{J}\left\{\frac{\partial^4 u}{\partial x^4}\right\} = \int_0^L \frac{\partial^4 u}{\partial x^4} X_n\, dx$$

which reduces to

$$\mathcal{J}\left\{\frac{\partial^4 u}{\partial x^4}\right\} = \int_0^L u\frac{d^4 X_n}{dx^4}\, dx = \lambda_n^4 \bar{u}$$

after performing multiple integration by parts, applying boundary conditions, and using the definition of the integral transform.

Solving PDE

Now that the integral transform has be constructed, the Euler-Bernoulli Equation 1 can be solved.

$$EI\frac{\partial^4 u}{\partial x^4} = -\rho\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} + S$$

First the integral transform $\mathcal{J}$ will be applied on the spacial x domain to get

$$EI\lambda^4 \bar{u}_n = -\rho\frac{\partial^2 \bar{u}}{\partial t^2} - c\frac{\partial \bar{u}}{\partial t}\bar{S}_n$$

Next the Laplace transform will be applied on the time domain with the initial conditions from equations 2

$$EI\lambda^4 \hat{\bar{u}}_n = -\rho s^2 \hat{\bar{u}}_n - cs\hat{\bar{u}}_n + \hat{\bar{S}}_n$$

Solving for transformed function yields

$$\hat{\bar{u}}_n = \frac{\hat{\bar{S}}_n}{\rho s^2 + cs + EI\lambda_n^4}$$

Applying the inverse Laplace transform gives the following convolution

$$\bar{u}_n = \bar{S}_n * \left(\frac{1}{\rho\omega_d} e^{-\frac{c}{2\rho}t} \sin\omega_d t\right)$$

where

$$\omega_d = \sqrt{\frac{EI\lambda_n^4}{\rho} - \left(\frac{c}{2\rho}\right)^2}$$

Then the inverse finite integral transform gives

$$u_n = \sum_{n=0}^{\infty} \left[\bar{S}_n * \left(\frac{1}{\rho\omega_d} e^{-\frac{c}{2\rho}t} \sin\omega_d t\right)\right] \frac{X_n(x)}{\|X_n\|^2} \quad (10)$$

where

$$\bar{S}_n = \mathcal{J}\{S(x,t)\} = \delta(t)X_n(x_p)$$

The solutions to this equation was coded and plotted in MATLAB. The script can be found in the appendix and also creates an animation file to visualize the vibration.

n	λ_n	ω_n
1	0	0
2	15.8	9.3
3	26.2	25.7
4	36.7	50.4
5	47.1	83.3
6	57.6	124.4

Table 1: First six eigenvalues, λ_n and natural frequencies, ω_n

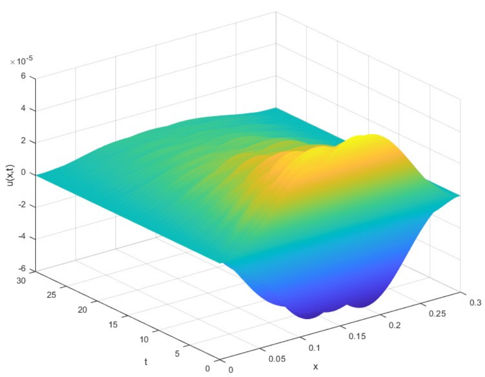

Figure 4: Orthogonal View of 3D plot of the displacement of the beam in time

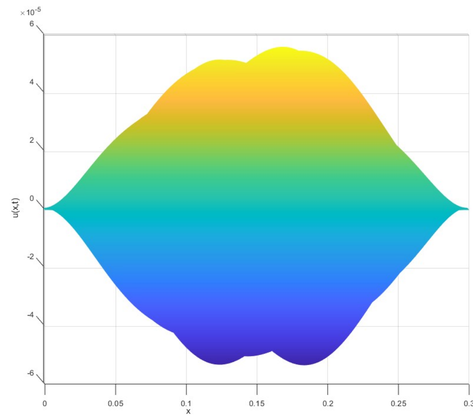

Figure 5: x-u view of 3D plot of the displacement, showing displacement along the length of the beam

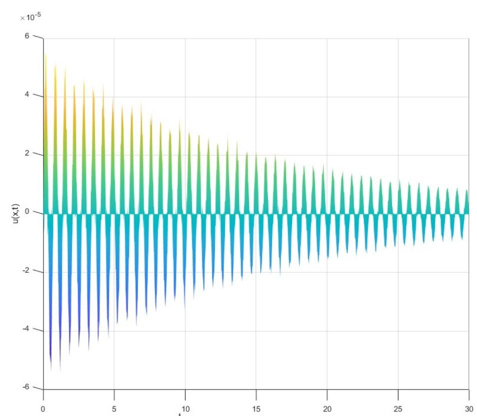

Figure 6: t-u view of 3D plot of the displacement, showing the damping of the displacement with respect to time

Results and Discussion

The first four eigenvalues of Equation 6 and corresponding natural frequencies are presented in Table 1. The first eigenvalue and natural fequency are both zero, which corresponds to rigid body motion. This outcome is expected since the system has one degree of freedom, allowing for translational motion without resistance.

The response of the beam is visualized in Figure 4, which plots the solution u_n from Equation 10 in both the time and spacial domains. The figure confirms that the solution satisfies the boundary conditions given in Equation 3, where both the deflection and slope are zero at the beam's ends. These boundary conditions are evident in the x-u plot in Figure 5, where the displacement is consistently zero at $x = 0$ and $x = L$.

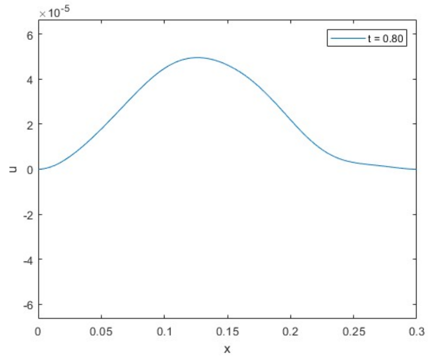

Figure 7: One instance in time of the displacement of the beam at t = 0.8

Damping effects are also captured in the response. As shown in Figure 6, the amplitude of the beam's displacement is initially large and gradually diminishes over time, consistent with the expected damping behavior.

The supplied code that is located in the Appendix also successfully provides an animated visualization of the plot of the response. One snapshot of the animation shows the beam's displacement profile at $t = 0.8$ can be seen in Figure 7. The animation also illustrates the beam's oscillatory behavior and confirms the damping effect, as the amplitude visibly decreases compared to earlier time steps.

Conclusions

This paper presents an analytical solution to the Euler-Bernoulli beam equation with damping for a system of two bolted beams. By leveraging the symmetry of the system and employing integral transform techniques, the study simplifies the complex dynamics of bolted connections into a manageable framework. The derived solution provides a deeper understanding of the damping effects on beam vibrations, laying a foundation for further exploration into bolted joint behavior.

This work serves to provide an analytical solution to a bolted system to gain more intuition and understanding to damping systems. Expanding the framework to account for more complex damping phenomena will enhance its applicability to broader structural and mechanical systems.

References

[1] Jonathan K. Black and Josh Blackham. "Solution of Euler-Bernoulli Beam Equation by Integral Transform Method". In: *Journal of Applied Engineering Mathematics* 10 (2023).

[2] Jerry H. Ginsberg. *Mechanical and Structural Vibrations*. 1st ed. 2001.

[3] Vladimir Solovjov. *Integrated Engineering Mathematics*. 1st ed. 2021.

Appendix

The MATLAB code for this project can be accessed here: `https://drive.google.com/drive/folders/1nI3IK2kM5uqBIjZIFRxPs5bdrnQkGcn4?usp=sharing`

