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Manning's and Navier-Stokes in Rectangular Channels

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Abstract

Manning's equation is widely used for estimating flow in open channels due to its simplicity and empirical basis. However, its accuracy can be limited in complex flow conditions. This study compares the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under varying Reynold's Numbers. Assuming a smooth channel (n = 0.01), we compared the two flowrates. Our results indicated that exact matches between the two equations occurent when the aspect ratio (α) of the channel was between 17 and 45, with errors less than 5%. This agreement occurred when the flow depth was significantly greater than the channel width, suggesting that under these conditions, the roughness assumptions of Manning's equation align more closely with the fundamental physics captured by the Navier-Stokes equations. These findings highlight the limitations of Manning's equation in laminar flow conditions and those of applying the simplified Navier-Stokes equations to turbulent flow.

Nomenclature

Navier-Stokes Variables:

$u_{i/j}$ = Velocity vector (m/s)

P = Pressure (Pa)

μ = Dynamic viscosity (Pa·s)

g_i = Gravitational acceleration vector (m/s^2)

t = Time (s)

$x_{i/j}$ = Spatial coordinate (m)

ρ = Fluid density (kg/m^3)

ν = Kinematic viscosity (m^2/s)

Manning's Equation Variables:

Q = Volumetric Flowrate (m^3/s)

A = Cross-sectional area of flow (m^2)

n = Manning's roughness coefficient

R_H = Hydraulic radius (m)

P_w = Wetted perimeter (m)

S = Channel slope (m/m)

V = Average velocity (m/s)

Shared Variables:

b = Channel bottom width (m)

h = Flow depth (m)

Re = Reynolds number (dimensionless)

α = h/b = Aspect ratio (dimensionless)

Introduction

Measuring and predicting flow in open channels is a fundamental aspect of both hydraulic engineering and engineering hydrology. Several methods exist for estimating flow characteristics, the most import of which being volumetric flow rate (Q).

The most commonly used method for estimating flow in open channels is Manning's equation:

$$Q = \frac{1}{n} A R_H^{2/3} S^{1/2}, \tag{1}$$

where hydraulic radius R_H is defined as the ratio of the cross-sectional area of flow to the wetted perimeter, given as

$$R_H = \frac{A}{P_w}. \tag{2}$$

Manning's equation is applied to all open channels, from culverts, streams, and rivers to large canals with gravity-driven flows [2]. It's popularity stems from its simplicity and, allowing engineers to quickly estimate flow rates based on channel characteristics [2]. However, Manning's equation has several limitations, including its purely empirical (rather than theoretical) nature, assumptions of uniform flow, and sensitivity to the roughness coefficient, n, which must be estimated for the channel [1]. These limitations can lead to inaccuracies in flow conditions that exhibit varying channel geometries, varying channel roughnesses, unsteady flows, and laminar flow regimes.

For more complex flow conditions, the Navier-Stokes equations provide a comprehensive framework for modeling fluid dynamics. The Navier-Stokes equations describe the motion of viscous fluid substances and are derived from the principles of conservation of mass, momentum, and energy. They can capture a wide range of flow phenomena, including turbulence, boundary layer effects, and non-uniform flow profiles.

In this paper, we evaluate the accuracy of Manning's equations for laminar flow conditions in an open rectangular channel (1). We do this by comparing volumetric flowrates obtained with Navier-Stokes equation for laminar flow to those obtained by Manning's equation for several flow velocities and channel aspect ratios. We do not solve the full Navier-Stokes equations, but rather simplify them using the conditions for Manning's equation, except for turbulence.

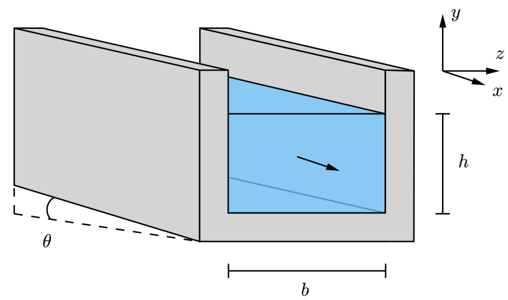

Figure 1: Rectangular channel and coordinate system.

Methodology

Governing Equations

We determine the flow regime using the Reynolds number (Re), defined as

$$Re = \frac{V R_H}{\nu}, \tag{3}$$

where V is the characteristic velocity (m/s), R_H is the hydraulic radius (m), and ν is the kinematic viscosity (m^2/s). We will consider laminar flow conditions to be such that $Re < 500$.

For gravity-driven flows, the Navier Stokes equation is given as

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho g_i, \tag{4}$$

where ρ is the fluid density (kg/m^3), u_i is the velocity vector (m/s), P is the pressure (Pa), μ is the dynamic viscosity (Pa·s), and g_i is the gravitational acceleration vector (m/s^2).

Assumptions

The following assumptions were made to simplify the Navier-Stokes to match the conditions necessary to apply

Manning's equation:

- The fluid is water $\rightarrow \rho = C, \mu = C$.
- Flow is steady $\rightarrow \frac{\partial}{\partial t} = 0$.
- Flow is uniform $\rightarrow \frac{\partial P}{\partial x} = 0$.
- Channel is infinite in the flow direction $\rightarrow \frac{\partial}{\partial x} = 0$.
- Flow is irrotational $\rightarrow \mathbf{u} \cdot \nabla \mathbf{u} = 0$.

Applying our assumptions to the x-momentum equation, we simplify it to

$$-\rho g_x = \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{5}$$

where g_x is the component of gravitational acceleration in the x-direction (m/s^2), and u is the velocity component in the x-direction (m/s).

With boundary conditions:

1. No slip at the bottom and sides:
   - $u|_{y=0,z} = 0$
   - $u|_{y,z=0} = 0$
   - $u|_{y,z=b} = 0$
2. Symmetry at the free surface:
   - $\frac{\partial u}{\partial y}|_{y=h,z} = 0$

These boundary conditions are illustrated in Figure 2.

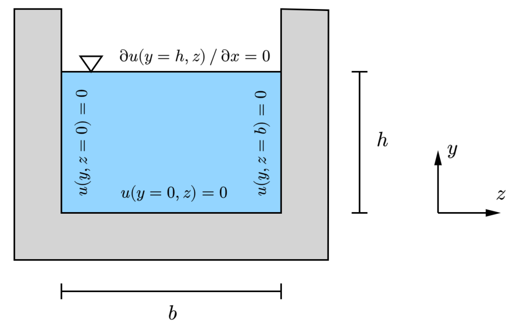

Figure 2: Boundary conditions for open channel flow.

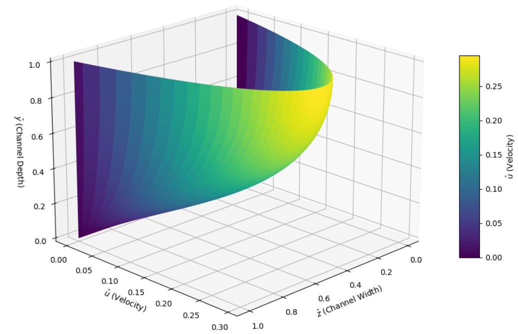

Figure 3: Non-dimensionalized velocity profile.

Velocity and Flowrate Calculations

Next we non-dimensionalize the equations using the following variables:

$$\hat{y} = \frac{y}{h}, \quad \hat{z} = \frac{z}{b}, \quad \hat{u} = \frac{u\nu}{h^2 g \sin\theta}, \tag{6}$$

where θ is the angle of the channel slope. Substituting these into the simplified Navier-Stokes equation, we obtain the non-dimensional form

$$-1 = \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \alpha^2 \frac{\partial^2 \hat{u}}{\partial \hat{z}^2}, \tag{7}$$

where $\alpha = \frac{h}{b}$ is the aspect ratio of the channel.

We solve this equation using separation of variables and apply the boundary conditions to find the velocity profile

$$\hat{u}(\hat{y}, \hat{z}) = \hat{y} - \frac{\hat{y}^2}{2} + \sum_{n=1}^{\infty} A_n \sin(\lambda_n \hat{y}) \cosh\left[\frac{\lambda_n}{\alpha} \left(\hat{z} - \frac{1}{2} \right) \right], \tag{8}$$

where

$$A_n = \frac{-2}{\lambda_n^3 \cosh\left(\frac{\lambda_n}{2\alpha}\right)} \quad \text{and} \quad \lambda_n = (2n-1)\frac{\pi}{2} \tag{9}$$

As shown in Figure 3, the flow distribution meets the boundary conditions defined previously.

The volumetric flow rate Q is calculated by integrating the velocity profile over the cross-sectional area:

$$Q = \int_0^b \int_0^h u(y,z)\,dy\,dz. \tag{10}$$

We are left with a final expression for Q in terms of channel dimensions and fluid properties:

$$Q = \frac{h^3 b g \sin\theta}{\nu}\left[\frac{1}{3} - \frac{4h}{b}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^5}\tanh\left(\frac{\lambda_n b}{2h}\right)\right] \tag{11}$$

Reynold's Number Selection

To evaluate the performance of Manning's equation under laminar flow conditions, we selected a range of Reynolds numbers (Re) from 250 (laminar) to 12,500 (fully turbulent).

For each selected Re, we calculated the flowrate for a combination of channel widths (b = 0.01 to 5 m) and slopes (S = 0.01 to 0.1 m/m). From the Navier-Stokes derived flowrate, we calculated Manning's n using the rearranged Manning's equation. If Manning's equation were accurate, the calculated n values would fall within typical ranges for smooth rectangular channels (0.009 to 0.012).

Results and Discussion

We found that no Re yielded similar discharge values in both the Navier-Stokes and Manning's equations. when using typical values of Manning's n for smooth rectangular channels. Figure 4 shows the comparison of flowrates across tested Re values, with small regions where the two flowrates are equivalent. Table 1 summarizes the best matches found for each Re, along with the corresponding aspect ratio

Table 1: Best Match per Reynolds Number

Re	Optimal α	Error (%)
250	34.1	0.0544
500	24.7	0.0167
2,000	23.2	0.593
12,500	24.1	0.0509

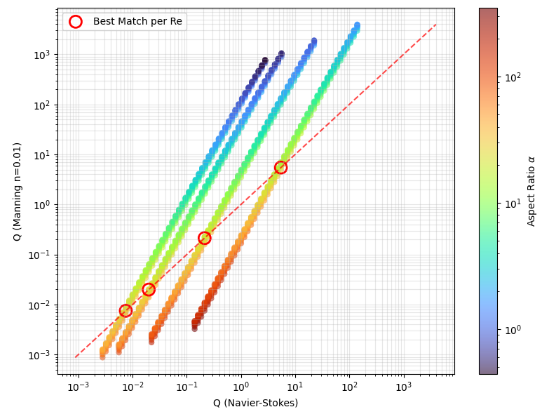

Figure 4: Comparison of Navier-Stokes and Manning's flowrates across Reynolds numbers.

Exploring further, we found a range of aspect ratios where the percent error between was minimized. Figure 5 shows the error between the two flowrates across aspect ratios for all tested Re values. The area highlighted in green indicates the range where the comparisons have an error of less than 5%.

Our analysis revealed the Navier-Stokes flowrates were greater than the Manning's flowrate for aspect ratios greater than about 45 and across all Re values. The opposite was true for aspect ratios less than 17. Table 2 summarizes the global aspect ratio range where the error between the two flowrates is less than 5%.

Table 2: Global α Range with $< 5.0\%$ Error

Statistic	Value
Min α	17.0
Max α	44.8
Mean α	26.7

The alpha values minimizing percent error suggest agreement occurs when the depth of flow is much greater than the width of the channel. Considering the fundamental physics governing the Navier-Stokes equations and the empirical nature of Manning's, we understanding why we see some agreement. Navier-Stokes account for the viscous forces acting across the entire cross-section of the

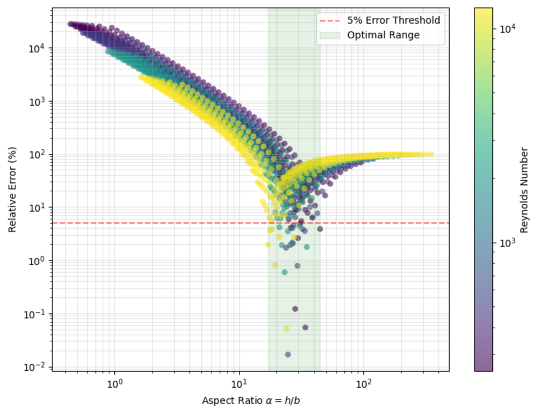

Figure 5: Error reduction of Navier-Stokes and Manning's Discharge.

flow, while Manning's simplifies these effects into a single roughness coefficient n. When the flow depth is significantly greater than the channel width, the influence of roughness on the channel walls is felt throughout the entire flow area. With roughness affecting the entire flow, the friction forces from Manning's equation behave more similarly to the viscous forces from the Navier-Stokes, leading to closer agreement in flowrates.

Conclusions

Our study compared the performance of Manning's equation with the Navier-Stokes equations in rectangular channels under laminar and turbulent flow conditions. We compared an analytically derived flowrate from the Navier-Stokes equations to the empirically derived flowrate from Manning's equation across a range of Re values (250 to 12,500). Our results indicated that exact matches between the two flowrates were not achievable across any Reynolds numbers using typical values of Manning's n for smooth channels. However, we identified a range of aspect ratios (17 to 45) where the error between the two flowrates was less than 5%. Agreement occured when the flow depth was significantly greater than the channel width, suggesting that the roughness assumptions of Manning's equation align more closely with the viscous forces captured by Navier-Stokes. These findings highlight the limitations of Manning's equation in laminar flow conditions and those of applying the simplified Navier-Stokes equations to turbulent flow.

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