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Acoustic Attenuation and Viscosity Estimation in Cylindrical Pipe

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ABSTRACT

This project develops a mathematical model describing the propagation of axisymmetric acoustic waves in a cylindrical pipe containing a viscous fluid. The model incorporates Bessel eigenfunctions arising from the separation of variables in cylindrical coordinates and includes exponential attenuation due to viscous dissipation. The resulting forward model relates viscosity μ to an attenuation coefficient α(μ). A simple inverse model is derived to estimate viscosity from noisy attenuation measurements. Numerical simulations and visualizations, including time-evolving pressure fields, viscosity-sweep fields, and 3D volumetric reconstructions are implemented in Python. The results demonstrate the mathematical structure of acoustic modes in cylindrical geometries and illustrate how viscosity affects wave attenuation and the accuracy of inverse parameter recovery.

Introduction

Acoustic sensing in cylindrical geometries arises in many engineering applications, including nondestructive evaluation, fluid property measurement, and industrial metrology. When an acoustic source excites a pipe, the resulting pressure field satisfies the linear acoustic wave equation. In viscous fluids, the wave amplitude decays exponentially in the axial direction, with the decay rate depending on viscosity.

The goals of this project are:

1. To construct a mathematical model for the pressure field $p(r, x, t)$ of an axisymmetric acoustic mode inside a cylindrical pipe.
2. To derive a simplified forward relation between viscosity μ and attenuation α.
3. To formulate and test an inverse problem for estimating viscosity from noisy measurements.
4. To implement numerical simulations and visualizations illustrating the mathematical behavior of the model.

This paper emphasizes the mathematical derivation and computational modeling, with physical interpretation provided where beneficial.

Mathematical Derivation of the Acoustic Field in a Cylindrical Pipe

This section derives the analytical expression used to model the acoustic pressure field $p(r, x, t)$ inside a cylindrical pipe filled with a viscous fluid. The derivation proceeds from the governing acoustic wave equation, through separation of variables in cylindrical coordinates, to the introduction of viscous attenuation and the final traveling-wave solution. This solution forms the basis for the numerical simulations and inverse viscosity estimation performed in this project.

For a small-amplitude acoustic disturbance in a homogenous fluid with no bulk flow, the linearized pressure field $p(x, t)$ satisfies the wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2},$$

where c is the speed of sound in the fluid. The geometry of interest is a cylindrical pipe, so it is natural to express the Laplacian in cylindrical coordinates (r, θ, x):

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}.$$

Axisymmetric pressure fields satisfy $\partial p/\partial \theta = 0$, yielding

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \quad (1)$$

A monochromatic acoustic excitation at angular frequency ω motivates a harmonic solution:

$$p(r, x, t) = \Re\{\tilde{p}(r, x)e^{-i\omega t}\}.$$

Substitution into (1) gives the spatial Helmholtz equation, with:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{p}}{\partial r}\right) + \frac{\partial^2 \tilde{p}}{\partial x^2} + k^2\tilde{p} = 0,\ k = \frac{\omega}{c}. \quad (2)$$

Assume a separable form:

$$\tilde{p}(r, x) = R(r)X(x).$$

Substituting into (2), dividing by R(r)X(x), and rearranging gives

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 = -\frac{1}{X}\frac{d^2 X}{dx^2}. \quad (3)$$

The left-hand side depends only on r, and the right-hand side only on x. Therefore, both sides must equal a separation constant, denoted $-\lambda^2$. This yields the coupled ODEs:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2 - \lambda^2)R = 0, \quad (4)$$

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0. \quad (5)$$

Equation (4) reduces to Bessel's equation of order zero:

$$r^2 R'' + rR' + (k_r^2 r^2)R = 0,\ k_r^2 = k^2 - \lambda^2.$$

The general solution is

$$R(r) = AJ_0(k_r r) + BY_0(k_r r),$$

Where J_0 and Y_0 are Bessel functions of the first and second kind. The second-kind solution $Y_0(k_r r)$ diverges as $r \to 0$, and therefore physical regularity on the pipe axis requires $B = 0$. The admissible solution is

$$R(r) = J_0(k_r r). \quad (6)$$

To determine allowable radial eigenvalues, a boundary condition is applied at the cylindrical wall. The acoustic field at the pipe wall is constrained by a Dirichlet boundary condition, enforcing zero acoustic pressure at $r = R$.

$$p(R, x, t) = 0 \rightarrow R(R) = 0.$$

This boundary condition gives the eigenvalue condition

$$J_0(k_r R) = 0.$$

The quantities $k_r R$ are therefore the zeros of the Bessel J_0 function. Selecting the fundamental axisymmetric mode corresponds to using the first zero.

$$k_r R = \lambda_1, \lambda_1 = 2.4048255577,$$

Thus, the radial mode shape

$$R(r) = J_0\left(\frac{\lambda_1 r}{R}\right). (7)$$

Equation (5) admits harmonic traveling-wave solutions:

$$X(x) = e^{\pm i\lambda x}.$$

In a lossless fluid, $\lambda = k = \omega/c$. A physically propagating real-valued pressure field is obtained by taking the cosine component:

$$X(x)T(t) = cos(kx - \omega t). (8)$$

In a viscous fluid, the acoustic wave develops a complex axial wavenumber:

$$k_{complex} = k + i\alpha,$$

Where $\alpha > 0$ is the attenuation coefficient, proportional to viscosity μ. The axial dependence then becomes

$$e^{i(k+i\alpha)x - i\omega t} = e^{-\alpha x} e^{i(kx-\omega t)}.$$

Taking the real part yields

$$X(x)T(t) = e^{-\alpha x} cos(kx - \omega t). (9)$$

In this project, the attenuation is assumed to be modeled by a proportional relation:

$$\alpha(\mu) = k_\mu \mu,$$

Combining the radial solution (7) with the viscously attenuated axial traveling wave (9), the final expression used throughout this project:

$$p(r, x, t; \mu) = J_0\left(\frac{\lambda_1 r}{R}\right) e^{-\alpha(\mu)x} cos(kx - \omega t). (10)$$

This closed-form analytical solution provides an efficient, physically based model for simulating acoustic propagation and attenuation inside a cylindrical pipe. All numerical field plots, animations, and inverse-viscosity estimates are generated directly from Equation (10).

Inverse Problem: Estimating Viscosity

The acoustic model shows that the pressure field has a form such that the amplitude envelope decays exponentially along the axial direction. The inverse estimation procedure relies only on the axial amplitude decay. The amplitude at position x is written as

$$A(x) = A_0 e^{-\alpha(\mu)x},$$

where $\alpha(\mu)$ denotes the attenuation coefficient associated with the fluid of viscosity μ and is unaffected by the choice of radial boundary conditions. Although the radial eigenfunction $J_0(\lambda_1 r/R)$ arises from a Dirichlet condition $p(R) = 0$, this assumption does not influence the form of the attenuation-based inversion. Thus, the Dirichlet boundary condition used in the forward model is mathematically consistent with, but not restrictive for, the inverse problem.

To estimate attenuation from experimental data, two sensors located at axial positions $x_1 < x_2$ are considered. Letting $A_1 = A(x_1)$ and $A_2 = A(x_2)$ denote the measured peak amplitudes at these locations. Using the exponential form,

$$A_1 = A_0 e^{-\alpha x_1}, A_2 = A_0 e^{-\alpha x_2},$$

and taking the ratio eliminates the initial amplitude A_0. Applying the natural logarithm gives an explicit expression for the attenuation coefficient:

$$\alpha = \frac{1}{x_2 - x_1} ln\left(\frac{A_1}{A_2}\right).$$

The inversion model represents the viscosity attenuation using the linear relation.

$$\alpha(\mu) = k_\mu \mu + \alpha_{wall},$$

where k_μ is a frequency-dependent proportionality constant and α_{wall} accounts for additional losses associated with the pipe boundary. Inverting this expression yields the viscosity estimate

$$\hat{\mu} = \frac{\alpha - \alpha_{wall}}{k_\mu}.$$

This inversion problem is algebraically simple but sensitive to noise in the measured amplitudes A_1 and A_2. Small fluctuations in the amplitude ratio are amplified by the logarithm, producing significant variability in the recovered viscosity. The resulting jitter in the animation demonstrates that the inverse problem is ill-conditioned: small measurement noise leads to large fluctuations in the reconstructed viscosity, even though the true viscosities vary smoothly. The extent of this sensitivity is illustrated in Figure 3.

Numerical Simulations

All numerical work was performed in Python using NumPy, SciPy, and Matplotlib. The pressure field is discretized on a grid in (r, x), and Bessel functions are evaluated using SciPy's j0() function.

Simulations include:

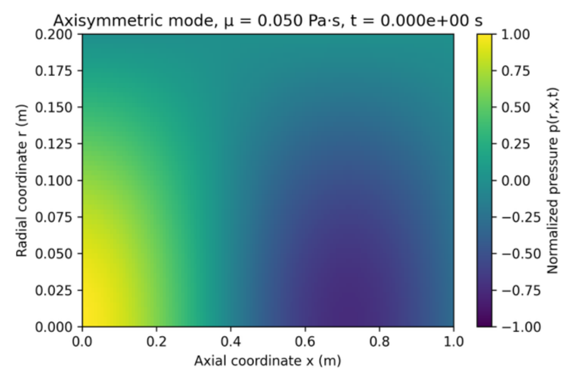

Figure 1: *Snapshot at t=0 of p(r, x, t) for a 1 m long pipe with fixed viscosity μ = 0.05 Pa·s.*

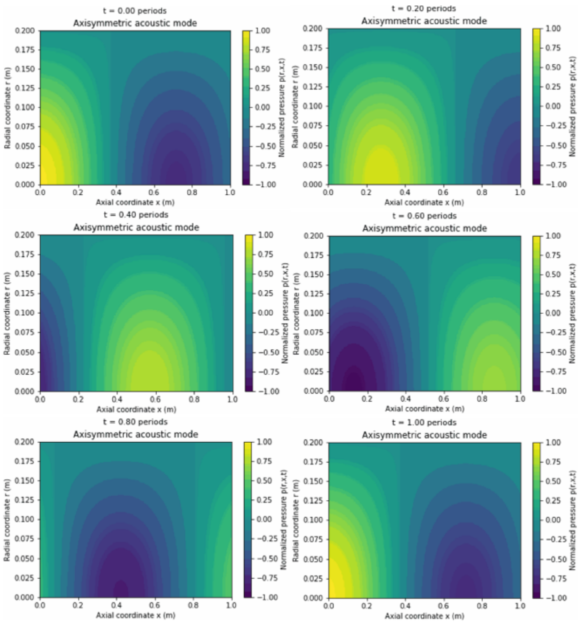

Figure 2: *Time animation of p(r, x, t) for a 1 m long pipe with fixed viscosity μ = 0.05 Pa·s through one period.*

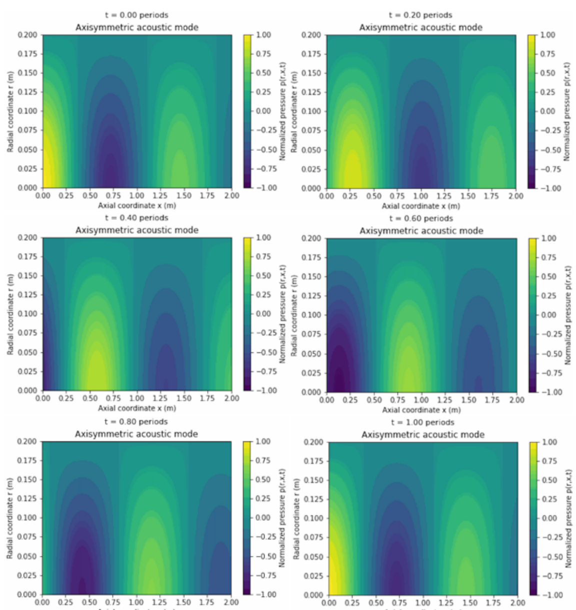

Figure 3: *Time animation of p(r, x, t) for a 2 m long pipe with fixed viscosity μ = 0.05 Pa·s through one period.*

This illustrates wave propagation and axial attenuation inside the pipe as time evolves.

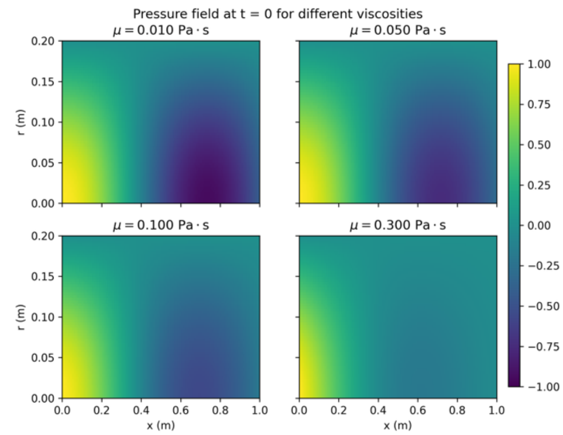

Figure 4: *Pressure field at t=0 for viscosities from 0.010-0.30 Pa·s*

The field decays more rapidly as viscosity increases.

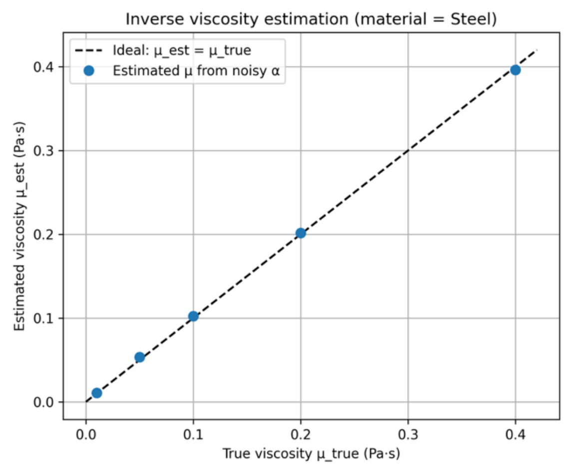

Figure 5: *Noisy inversion of viscosity showing jitter in μ.*

The scatter around the line $\mu_{est} = \mu_{true}$ quantifies sensitivity.

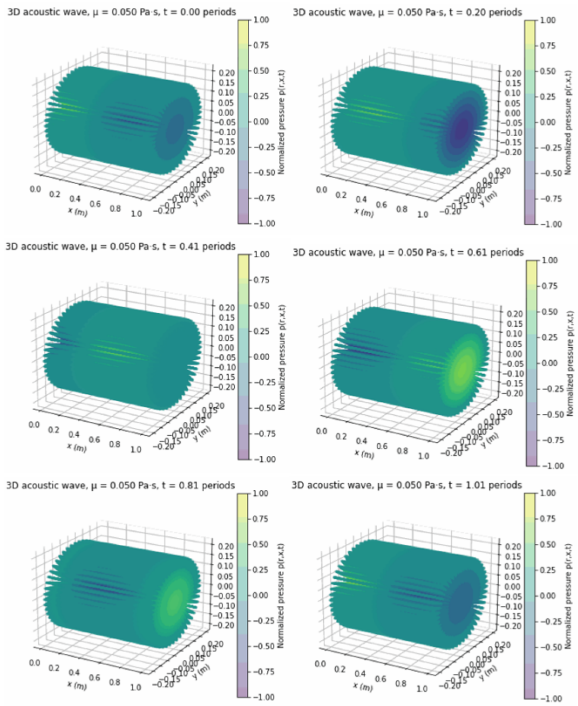

Figure 6: *3D rendering of the full pressure field through one period.*

Shows the radial structure of J_0 and axial attenuation as time evolves.

Discussion

The project demonstrates how classical mathematical tools, Bessel functions, separation of variables, and exponential decay models accurately capture acoustic behavior in cylindrical geometries. The radial structure of the fundamental mode arises directly from the Bessel eigenvalue problem. The axial structure reflects a harmonic traveling wave with viscosity-dependent damping.

The inverse problem highlights a key mathematical issue: linear inversion amplifies measurement noise. Even 5% noise in attenuation can yield much larger variance in viscosity estimates. This behavior is typical of ill-conditioned linear inverse problems and suggests a need for regularization or multi-sensor averaging in practical systems.

The visualizations confirm the model's behavior: low viscosity preserves wave amplitude over long distances, whereas higher viscosity damps the field quickly, and the 3D renderings illustrate the symmetry and structure of the solution.

Conclusion

This project applied mathematical modeling techniques to derive, analyze, and simulate acoustic wave behavior in cylindrical fluid-filled pipes. By solving the cylindrical Bessel eigenproblem and incorporating viscous attenuation, a compact analytical solution was obtained. Numerical simulations validated the model and provided insight into both forward wave dynamics and inverse-viscosity estimation.

The project demonstrates the power of classical mathematical methods in understanding complex physical systems and provides a foundation for more advanced inverse and forward modeling of acoustic sensing in fluids.

References

[1] Morse, P.M., & Ingard, K. U. *Theoretical Acoustics*. Princeton University Press. 1987.

[2] SciPy Documentation: `scipy.special.j0`.

[3] Vladimir Solovjev. *Integrated Engineering Mathematics*. 2025.

APPENDIX

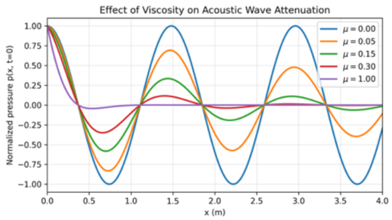

Figure 7: *1D wave attenuation for various viscosities.*

A one-dimensional model is performed with a modified equation that lacks the radial aspect.

$$p(x,t;\mu) = e^{-\alpha(\mu)x}cos(kx - \omega t).$$

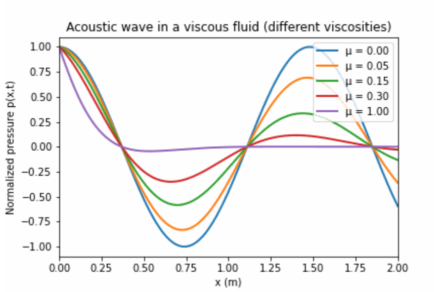

Figure 8: *1D wave attenuation for various viscosities.*

Code:

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Mon Dec 15 16:41:18 2025

@author: connermurray

Cylindrical acoustic model + viscosity-attenuation curves
and inverse estimation of viscosity.
"""


import numpy as np
from scipy.special import j0

# ------------------------------
# Cylindrical acoustic model
# ------------------------------
# p(r, x, t; mu) = exp(-alpha x) * J0(lambda1 * r / R) * cos(k x - omega t)
# alpha = k_mu * mu

# Physical-ish parameters
c = 1480.0          # speed of sound in fluid (m/s)
f = 1000.0          # excitation frequency (Hz)
omega = 2 * np.pi * f
k = omega / c       # axial wavenumber
k_mu = 8.0          # proportionality factor linking viscosity to attenuation

# Pipe geometry
L = 1.0             # pipe length (m)
R = 0.2             # pipe radius (m)

# Fluid properties (e.g. water-like)
rho_f = 1000.0                      # kg/m^3
c_f = 1500.0                        # m/s
Z_f = rho_f * c_f                   # acoustic impedance of fluid

# Discretization in r and x
nx = 120
nr = 60
x = np.linspace(0, L, nx)
r = np.linspace(0, R, nr)

# 2D grids: rows are r, columns are x
R_grid, X_grid = np.meshgrid(r, x, indexing="ij")

# Wall materials (density & sound speed)
materials = {
    "Steel":    {"rho": 7850.0, "c": 5900.0},
    "Aluminum": {"rho": 2700.0, "c": 6400.0},
    "Titanium": {"rho": 4500.0, "c": 6100.0},
```

1

```
    "PVC":      {"rho": 1400.0, "c": 2300.0},
}

# Base scaling for wall-related loss
k_wall_base = 1.0   # arbitrary 1/m

# Viscosity range for curves (Pa·s)
mu_vals = np.linspace(0.001, 0.5, 200)

# Time settings for animation
n_periods = 4.0
T = n_periods / f       # total duration shown (~4 periods)


fps = 60
frames_per_period = 60
nt = int(n_periods * frames_per_period)
times = np.linspace(0, T, nt, endpoint = False)


def wall_attenuation(Z_w):
    """
    Wall attenuation from impedance mismatch.
    Plane-wave normal-incidence transmission:
        T = 4 Zf Zw / (Zf + Zw)^2
    Use (1 - T) as a simple 'loss' factor and scale it.
    """
    T = 4 * Z_f * Z_w / (Z_f + Z_w)**2
    return k_wall_base * (1.0 - T)


# Precompute alpha_wall(material)
alpha_wall_mat = {}
for name, props in materials.items():
    Z_w = props["rho"] * props["c"]
    alpha_wall_mat[name] = wall_attenuation(Z_w)


def alpha_fluid(mu):
    """Fluid attenuation from viscosity."""
    return k_mu * mu


def alpha_total(mu, material):
    """Total attenuation for given viscosity and wall material."""
    return alpha_fluid(mu) + alpha_wall_mat[material]


def estimate_mu_from_alpha(alpha_meas, material):
    """
```

2

```
    Inverse relation:
        alpha_meas = k_mu * mu + alpha_wall(material)
        => mu_est = (alpha_meas - alpha_wall) / k_mu
    """
    return (alpha_meas - alpha_wall_mat[material]) / k_mu


def J0_series(z, n_terms=12):
    """
    Approximate Bessel function J0 using the power series:
    J0(z) = sum_{m=0}^\infty (-1)^m (z^2/4)^m / (m!)^2
    DEFUNCT
    """
    result = np.zeros_like(z)
    z2_over4 = (z**2) / 4.0
    term = np.ones_like(z)
    result += term   # m = 0 term
    for m in range(1, n_terms):
        term *= -z2_over4 / (m * m)
        result += term
    return result


def pressure_field(mu_scalar, Rg, Xg, t):
    """
    Axisymmetric acoustic field in cylindrical coordinates.
    mu_scalar should be a single viscosity value (float),
    not an array.
    """
    alpha = k_mu * mu_scalar
    lambda1 = 2.404825557695773  # first root of J0

    radial = j0(lambda1 * Rg / R)
    # radial = J0_series(lambda1 * Rg / R)
    axial = np.exp(-alpha * Xg) * np.cos(k * Xg - omega * t)
    return radial * axial
```

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