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MATHEMATICAL MODEL OF A BOUNCING BALL

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Abstract

A bouncing ball exhibits interesting vibrations, especially when it has a high flexibility. This study seeks to model the behavior of a bouncing ball using the wave equation and the circular integral transform method. To solve this problem, the solution for the behavior of vibrations in a circular string was found, and then, assuming symmetry, it was projected in a circular pattern to form a sphere. The resulting equations were then plotted using mathematical software, and it was found that, while imperfect, this simplified model captures some of the complex behavior of a bouncing ball.

Nomenclature

Wave equation, circular integral transform, periodic continuity

Introduction

A bouncing ball experiences complex elastic deformations that can be difficult to model as the deformation propagates around the ball.

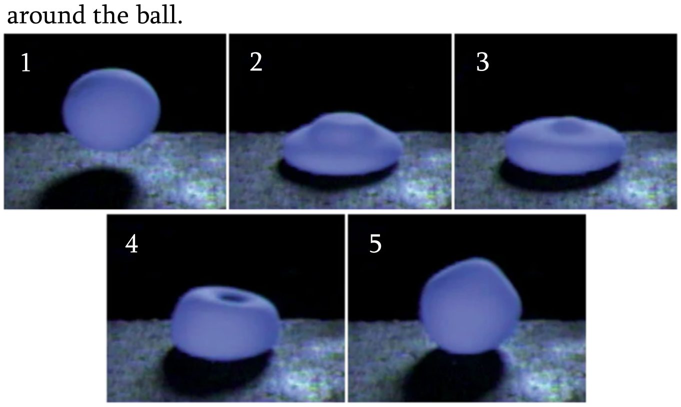

Figure 1: Images of a racquetball bouncing off a surface captured with a slow-motion camera [1].

This study seeks to determine what aspects of this behavior can be captured using a solution to the wave equation, and what initial conditions can be used to best model the impact of a ball on a surface like the one shown in Figure 1.

Theory

The circular integral transform is an application of the Fourier Series in the interval $[0,2\pi]$ that has periodic continuity, meaning:

$$u(\theta) = u(\theta + 2\pi) \tag{1}$$

$$\frac{\partial}{\partial\theta}u(\theta) = \frac{\partial}{\partial\theta}u(\theta + 2\pi) \tag{2}$$

In this study, the circular integral transform was applied to Equation 3, which represents the vibrations of a circular string. The solution was then projected in a circular pattern to model a ball.

$$\frac{\nu^2}{R^2}\frac{\partial^2 u}{\partial\theta^2} = \left(\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t}\right) \tag{3}$$

In this equation, which we will refer to as h later in the paper, R is the radius of the circle, ν is a constant representing the stiffness/flexibility of the string, and γ represents the damping of the system. The variable ν is a constant representing the properties of the string (or ball, depending on the application). One study, [2] calculated ν by taking the square root of the tension in the string divided by the mass density of the string:

$$\nu = \sqrt{T/\mu} \tag{4}$$

Since these properties were unknown for this study, simulations were used to find an ideal values for ν and γ.

Initial conditions are described by:

$$u(\theta, 0) = f(\theta) \tag{5}$$

$$\frac{du}{d\theta}(\theta, 0) = 0 \tag{6}$$

Applying the Circular Integral transform to this equation and the initial conditions yields the following:

$$\Phi_0\{h\} = 0 = \frac{\partial^2 \hat{u}_0}{\partial t^2} + 2\gamma \frac{\partial \hat{u}_0}{\partial t} \tag{7}$$

$$\Phi_n\{h\} = \frac{-n^2\nu^2}{R^2}\hat{u}_n = \frac{\partial^2 \hat{u}_n}{\partial t^2} + 2\gamma \frac{\partial \hat{u}_n}{\partial t} \tag{8}$$

$$\hat{f}_n(\theta) = \int_0^{2\pi} f(s)ds \tag{9}$$

The characteristic equation resulting from Equation 3 yields

$$\hat{u}_0 = c_1 + c_2 e^{-2\gamma t} \tag{10}$$

The unknown constants depend on the initial condition equations. In this case, $\hat{u}_0$ is a constant that depends only on the initial deformation, $f(\theta)$, since the initial velocity is 0.

To solve for $\hat{u}_n$, a Laplace transform can be applied to Equation 7 which results in the following relationship:

$$\hat{U}_n = \frac{\hat{f}_n(\theta) * (s+1)}{(s+\gamma)^2 + \frac{n^2\nu^2}{R^2} - \gamma^2} \tag{11}$$

Where $\hat{f}_n(\theta)$ is the Circular Integral Transform of the initial condition equation for displacement. Equation 11 can then be rearranged based on the initial conditions present so that the inverse Laplace transform can be performed. This yields the following equation for $\hat{u}_n(\theta, t)$:

$$\hat{u}_n(\theta, t) = \hat{f}_n(\theta) e^{-\gamma t}\left[\cos(bt) + \tfrac{1-\gamma}{b}\sin(bt)\right] \tag{12}$$

Where

$$b = \sqrt{\frac{n^2\nu^2}{R^2} - \gamma^2} \tag{13}$$

Afterwards, the inverse circular inverse transform can be applied to solve for the equation $u(\theta, t)$. To relate this solution, which represents the behavior of a circular string, to a ball or sphere, the solution is projected in a circular pattern. The solution with any desired initial condition can be solved for using this method and then can be plotted using mathematical software (see appendix). The final solution is given by the following equation.

$$u(\theta, t) = \frac{1}{2\pi}\hat{u}_0 + \frac{1}{\pi}\sum_{n=1}^{\infty} \hat{u}_n(\theta, t) \tag{14}$$

Results

For this study, an initial displacement was applied to the ball, approximating its deformation when hitting a surface. Various initial conditions for the were analyzed to determine which could be used to best fit the behavior of a bouncing ball given the solution method described above. These initial conditions included two variations of Heaviside functions and a Dirac Delta function. These initial conditions are shown in Equations 10 through 12 respectively.

$$f_1(\theta, 0) = 0.004(.05 - \theta)H(\theta - 0.05) \tag{15}$$

$$f_2(\theta, 0) = 0.2\big(1 - H(\theta - 0.1)\big) \tag{16}$$

$$f_3(\theta, 0) = \frac{0.01\delta(\theta)}{2\pi} \tag{17}$$

The solutions based on each of these conditions were plotted using MATLAB and the values for ν and γ were optimized to find a combination that best matched the behavior observed in [1]. The values that were used are $\nu = 3$ and $\gamma = 5$.

In each case, the response lasted for about 0.5 seconds before the damping reduced the response to near zero deformation. This seems like a reasonable amount of time for the system that we are trying to simulate. Varying γ had a significant impact on the time of the response with large values damping out the system almost immediately. Variations in ν also had a significant influence on the response with high values causing higher vibration frequencies and longer response times.

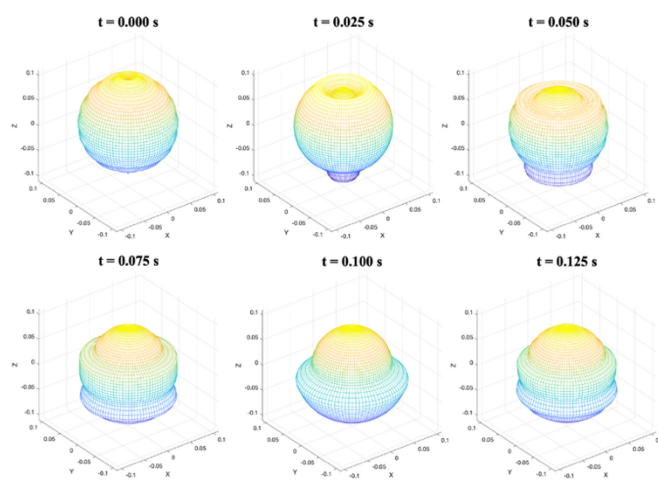

Figure 2: Response resulting from the initial condition using the Linear Heaviside function as described in Equation 15.

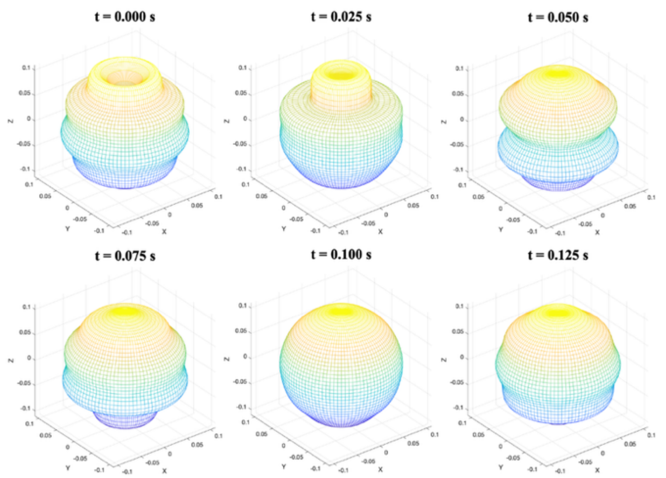

Figure 3: Response resulting from the initial condition using the Heaviside function as described in Equation 16.

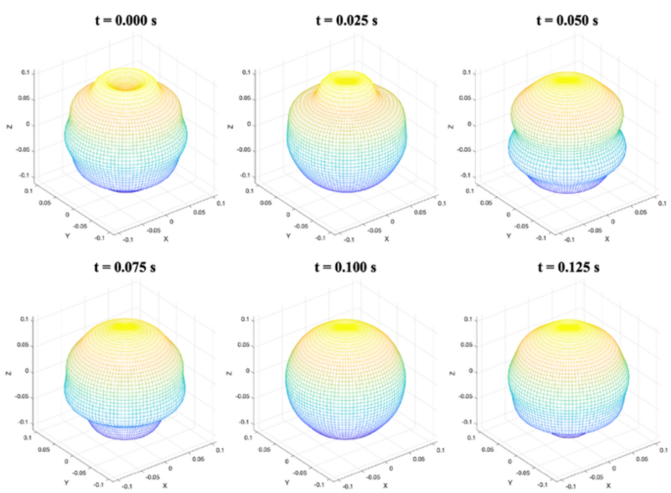

Figure 4: Response resulting from the initial condition using the Dirac Delta function as described in Equation 17.

Figures 2 through 4 show the simulated responses for each of the three Dirichlet initial condition equations at several time intervals during each response.

While none of the responses with these initial conditions match the exact behavior of the deformation of a ball bouncing off a surface, they do have many similar characteristics and behavior. Of the three boundary conditions we simulated, it appears that the Dirac Delta boundary condition (Equation 17) was the closest match to the physical behavior of the bouncing ball that is shown in Figure 1. However, more data representing the vibrations of a physical ball under impact would be needed for a more thorough comparison and analysis.

Conclusion

While the solutions to the wave equation shown here seem to behave similarly to a bouncing ball, they do not fully take into consideration the complexities involved in modeling the dynamics. Since the solution is based on the equation of a string, projected in a circular pattern, it does not consider effects of the thickness of the ball. The value for ν considers some of the material properties, but with only one parameter, the solution cannot fully describe the dynamics of the system which depend on many factors, not only the properties of the ball itself, but also the properties and geometry of the surface(s) it may be interacting with.

This solution could also be applied to other scenarios where vibrations travel around the surface of a sphere. Future work could be done to see how the behaviors seen in this study compare to bubbles, water droplets, balloons, and different types of balls with varying properties.

Additional work could also be done to simulate impacts of two balls colliding with each other as well as testing more complex initial conditions including initial velocities to find a better approximation of the real-world response.

Of the tested initial conditions, we found the Dirac Delta function (see Figure 4) to be the closest match to the physical behavior of a bouncing ball like the one shown in Figure 1. Other future work could include more comprehensive testing of types of functions for the initial displacement and/or velocity of the sphere to find what best approximates the behavior of the ball.

Acknowledgements

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