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Cooling of Hot Drink in a Cylindrical Cup

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Abstract

Hot drinks are a staple of daily life across cultures, yet their consumption is often delayed while they cool to a safe temperature. This study models the cooling of a hot beverage within a standard cylindrical cup by solving the heat conduction equation in cylindrical coordinates. The model predicts a cooling time from 90°C to a drinkable 60°C of approximately 70 minutes. This result, however, significantly exceeds common experiential cooling times of 10–20 minutes, indicating critical oversimplifications in the model. Future work must incorporate a composite wall model, include evaporative and radiative boundary conditions, and validate predictions against experimental data to achieve an accurate, physically representative cooling profile.

Nomenclature

- r_i cup radius
- T_i initial temperature
- T_a ambient temperature
- α Thermal diffusivity of water
- h Convective heat transfer coefficient for still air
- k Thermal conductivity of water

Introduction

The consumption of hot beverages—coffee, tea, and cocoa—is a deeply entrenched ritual in daily life across the globe, serving functions that range from physiological stimulant to social lubricant. Despite the enduring popularity of these drinks, a universal and practical challenge persists: they are initially served at temperatures near 90°C, far hotter than the preferred drinking temperature of approximately 60°C [1].

From an engineering perspective, predicting this cooling time is a classical transient heat transfer problem. The system involves conduction within the liquid, convective and radiative losses from its surfaces, and the significant insulating effect of the container itself. In this study, a preliminary approach is taken by treating the beverage as a long cylinder of liquid with angular symmetry.

Method

Start with the corresponding heat equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \tag{1}$$

The initial condition:

$$u(r,0) = T_i \tag{2}$$

Figure 1: A cup of hot milk.

For the boundaries conditions, the temperature in the center of the cup should be bounded. In addition, the heat transfer from the cup wall to the environment and the heat transfer from the hot drink to the cup wall should equal each other, assuming there is no radiation and contact resistance. The corresponding mathematical expressions are:

$$u(0,t) < \infty \tag{3}$$

$$-k\frac{\partial u}{\partial r}|_{r=r_i} = h[u(r_i,t) - T_a] \tag{4}$$

Let

$$v(r,t) = u(r,t) - T_a \tag{5}$$

and its corresponding boundary and initial conditions:

$$v(r,0) = T_i - T_a \tag{6}$$

$$v(0,t) < \infty \tag{7}$$

$$-k\frac{\partial v}{\partial r}|_{r=r_i} = hv(r_i,t) \tag{8}$$

Due to the existence of the Robbin condition, it is easier To solve $v(r,t)$ instead of $u(r,t)$ directly. $v(r,t)$ satisfy the same heat equation in equation(1). Now proceed to solve $v(r,t)$ with the method of the separation of variables, assuming $v(r,t) = R(r)T(t)$. Now Equation (1) becomes:

$$R''T + \frac{1}{r}R'T = \frac{1}{\alpha}RT' \tag{9}$$

Separate the variables and rearrange to get:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu \tag{10}$$

The partial differential equation(PDE) becomes two ordinary differential equations(ODE). First solve for R, notice it is a Bessel equation of order 0:

$$r^2R'' + rR' + (\lambda^2r^2 - 0^2)R = 0, \quad \mu = -\lambda^2 \tag{11}$$

The general solution for the Bessel equation is:

$$R(r) = C_1J_0(\lambda r) + C_2Y_0(\lambda r) \tag{12}$$

Since $v(r,t)$ is bounded at $r = 0$ and $Y_0(\lambda r)$ is unbounded at $r = 0$, $C_2 = 0$. Now apply the Robbin condition:

$$-kR'(r_i) = hR(r_i) \tag{13}$$

With the Bessel function identity $J_0'(x) = -J_1(x)$, Equation (12) becomes:

$$hJ_0(\lambda_n r_i) - k\lambda_n J_1(\lambda_n r_i) = 0 \tag{14}$$

The subscript n means there are multiple $\lambda's$ that satisfy this equation. The solution for R is shown below with λ_n satisfying Equation (13):

$$R_n(r) = J_0(\lambda_n r) \tag{15}$$

The next step is to solve for T:

$$T' + \alpha\lambda_n^2T = 0 \quad \mu = -\lambda_n^2 \tag{16}$$

Solves Equation (15) with characteristic equation gives the solution:

$$T_n(t) = e^{-\alpha\lambda_n^2 t} \tag{17}$$

Combine $R(r)$ and $T(t)$ to the total solution for $v(r,t)$:

$$v(r,t) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) e^{\alpha \lambda_n^2 t} \tag{18}$$

$$a_n = \frac{\int_0^{r_i} (T_i - T_a) J_0(\lambda_n r) \ r dr}{\int_0^{r_i} [J_0(\lambda_n r)]^2 \ r dr} \tag{19}$$

The final solution now is obtained from Equation (5) and (18):

$$u(r,t) = v(r,t) + T_a \tag{20}$$

Result

The following constants were used to visualize the result:

- $r_i = 0.04 \ m$
- T_i = 90 °C
- T_a = 25 °C
- $\alpha = 1.43 * 10^{-7} \ m^2/s$
- $h = 12 \ W/(m^2 K)$
- $k = 0.67 \ W/(mK)$

The code that generates the plots and animation is included in the appendix. Figure 2 shows it takes 70 minutes for the overall temperature of the drink to be 70 °C. Figure 3 plots Temperature against time, which shows it takes more than 100 minutes for the center of the drink to cool down to 60 °C.

Conclusions

The result obtained is far too long compared to real-world experience(10-20 minutes). The discrepancy is mostly due to the oversimplification of the model and a more sophisticated model needs to be developed in the future to improve the accuracy of the result. For reference, below are several additions to the model but were not applied to the current model due to the time constraint :

- Cup wall. The current model assumes water is in direct contact with air, which allows the liquid at surface cool instantly while leaving the inside of the liquid hot, creating an insulating shell.
- Evaporation. Evaporation carries away massive latent heat(2258 kJ/kg) and can be modeled as an additional heat boundary condition at the top surface.
- Thermal radiation, heat conduction at the bottom, finite height instead of long cylinder.

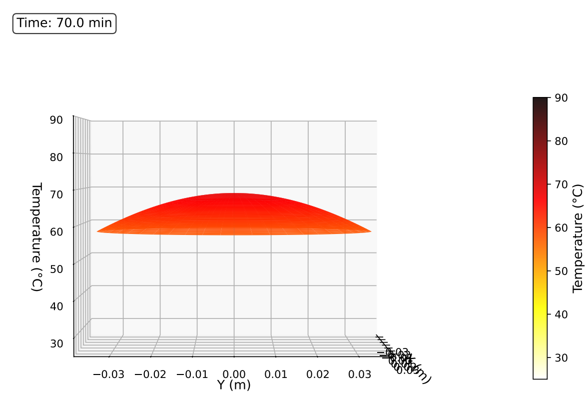

Figure 2: Temperature profile of the hot drink at 70 minutes.

Reference

[1] H.S. Lee and O'Mahony, *"At What Temperature Do Consumers Like to Drink Coffee?: Mixing Methods"*, Vol. 67, Issue 7, Journal of Food Science.

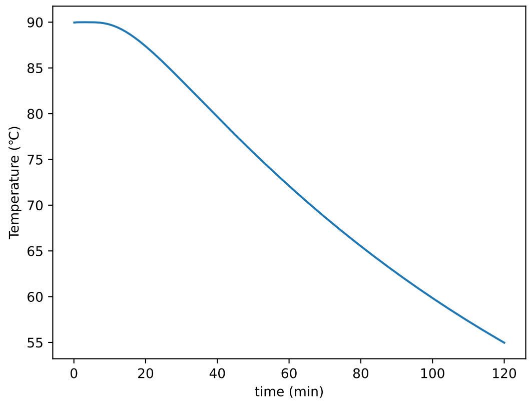

Figure 3: Temperature-Time profile at the center of the cup

Appendix

The Github link has the code for plot and animation generation:
Modeling a cup of hot drink in a cylindrical cup

