Journal of Applied Engineering Mathematics

Volume 12, December 2025

Modeling Seismic Activity due to Cosmic Impacts with the Circular Integral Transform

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Abstract

Seismic activity has far-reaching consequences for building safety and the environment. This paper presents a simplified model for seismic waves induced by cosmic impacts. This model applies the circular transform to solve the wave equation for a single radial coordinate with a localized initial displacement. Displacements and wave movements are simulated. Results are verified to match expected behavior. Simulations with damping representative of various geomaterials are conducted and compared. Implications and possible improvements to the model are discussed.

Keywords: Circular Integral Transform, Seismic Waves, Laplace Transform

1 Intro

Understanding and predicting consequences of seismic events is critical for understanding survivability of structures and making related design decisions. One of the earthquake waves that has the largest impact in structure survivability are Rayleigh waves (transverse waves). Survivability becomes a large concern when the seismic event becomes significant on the world scale, like a species-ending impact from an asteroid. In this paper, an analytical model for a seismic wave due to an asteroid impact is created using the circular integral transform and Laplace transform of a damped wave equation. The initial impact of the asteroid is modeled as a large displacement with a extremely small radius. The model is also tuned with real world parameters from previous studies to produce reasonable results. The simulation is rerun for different attenuation factors to see how different soil types affect the predicted results. The results presented here can help inform engineers where to build structures that will survive an extra-terrestrial impact—even if the engineers don't.

2 Methods

2.1 Assumptions and Initial Equation

The initial impact of the asteroid is assumed to have a small impact area, but introduce a significant displacement relative to the size of the earth. The crust of the earth is modeled as a very thin surface of a perfect sphere. The damping and other parameters of the system are assumed to be linear and homogeneous across the surface of the planet. The seismic waves of the earth's crust were assumed to have azimuthal symmetry about the impact point. Although significant, water is neglected in the solution. Additionally, the model assumes no mass loss due to ejection of material into space from the impact.

Given these assumptions, said impact can be modeled using the wave equation with one radial coordinate and a single linear damping term as seen in Equation 1. In this equation, u is the displacement, t refers to time, θ is the angular coordinate, R refers to the radius of the sphere, μ is the wavespeed of the surface, and γ relates to the damping of the material. For this paper, the collision is modeled as an initial displacement as seen in equations 2 and 3.

$$\frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\mu^2}\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} \tag{1}$$

$$u(\theta,0) = \frac{M_0\delta(\theta)}{2\pi R} \tag{2}$$

$$\frac{\partial u}{\partial t}(\theta,0) = 0 \tag{3}$$

$$u(\theta,t) = u(\theta+2\pi,t) \tag{4}$$

$$\frac{\partial u}{\partial t}(\theta,t) = \frac{\partial u}{\partial t}(\theta+2\pi,t) \tag{5}$$

2.2 Circular Transform Method

Given the radial coordinate system and recurring boundary conditions, Equations 4 and 5, the circular integral transform is an appropriate solution method for this form of the wave equation. Applying the circular integral transform to equations 1 through 3 yields the transformed differential equation 6 with initial conditions 7 and 8.

$$\frac{-n^2}{R^2}u_n = \frac{1}{\mu^2}\frac{\partial^2 u_n}{\partial t^2} + 2\gamma\frac{\partial u_n}{\partial t} \tag{6}$$

$$\hat{u}_n(\theta,0) = M_0\cos(n\theta) \tag{7}$$

$$\frac{\partial \hat{u}_n}{\partial t}(\theta,0) = 0 \tag{8}$$

Equation 6 simplifies to Equation 9, a linear, homogeneous, second-order differential equation.

$$\frac{\partial^2 u_n}{\partial t^2} + 2\gamma\mu^2\frac{\partial u_n}{\partial t} + \frac{n^2\mu^2}{R^2}u_n = 0 \tag{9}$$

This type of ODE can be solved using the Laplace transform. Transforming each term from variable $\hat{u}_n(\theta,t)$ to $U_n(\theta,s)$and simplifying, we arrive at Equation 10

$$U_n(\theta,s) = M_0cos(n\theta)\frac{s+2\gamma\mu^2}{s^2+2\gamma\mu^2 s+n^2\mu^2/R^2} \tag{10}$$

Using the quadratic formula, we can factor the denominator to arrive at a form convenient for performing the inverse Laplace transform.

$$U_n(\theta,s) = M_0cos(n\theta)\frac{s+2\gamma\mu^2}{(s-r_1)(s-r_2)} \tag{11}$$

where $r_{1,2}$ are defined as:

$$r_{1,2} = -\gamma\mu^2 \pm \sqrt{\gamma^2\mu^4 - n^2\mu^2/R^2} \tag{12}$$

The inverse laplace results in the solution:

$$\hat{u}_n = \frac{M_0cos(n\theta)}{r_1-r_2}[(2\gamma\mu^2+r_1)e^{r_1 t} - (2\gamma\mu^2+r_2)e^{r_2 t}] \tag{13}$$

Unfortunately, when the roots, $r_{1,2}$ are complex, this solution outputs a complex result with no physical meaning. Therefore, another inverse Laplace approach is taken for complex-valued roots, which yields a real-valued result.

To make the derivation of the complex root solution legible, we first use the following substitutions to Equation 10: $a = \gamma\mu^2$, $\omega^2 = n^2\mu^2/R^2$, and $K = M_0cos(n\theta)$. Completing the square in the denominator, and separating the numerator results in

$$U(\theta,s) = K\Bigg[\frac{s+a}{(s+a)^2+(w^2-a^2)} + \frac{a}{\sqrt{w^2-a^2}}\frac{\sqrt{w^2-a^2}}{(s+a)^2+(w^2-a^2)}\Bigg] \tag{14}$$

Performing the inverse Laplace transform then yields Equation 15

$$\hat{u}_n = K\Bigg[e^{-at}\cos\Big(t\sqrt{\omega^2-a^2}\Big) + \frac{a}{\sqrt{\omega^2-a^2}}e^{-at}\sin\Big(t\sqrt{\omega^2-a^2}\Big)\Bigg] \tag{15}$$

The final solution shown in Equation 16 consists of an infinite series of these solutions, where the equation used for $\hat{u}_n$ depends on whether the roots of the characteristic polynomial are real or complex. If the roots are real valued for a given value of n, Equation 13 is used, and if the roots are complex, Equation 15 is used.

$$u = \frac{1}{2\pi}\hat{u}_0 + \frac{1}{\pi}\sum_{1}^{\infty}\hat{u}_n \tag{16}$$

3 Results

The parameter values used in our solution are given in Table 1. For R, the mean diameter of the earth was used. For μ and γ, values were derived from [1], [2], and [3]. The Q values given in these articles were converted to an equivalent value for γ using Equation 17 and assuming $f = .005$. Furthermore, our Fourier series solution was truncated at $n = 50$.

$$\gamma = \frac{\pi f}{Q\mu^2} \tag{17}$$

Table 1: Model Parameters for nominal and alternative cases

Parameter	Earth Crust (Nominal)	Sandstone	Soda-Lime Glass
M_0 (m)	.0	.0	.0
μ (m/s)	4500	4500	4500
γ (s/m^2)	$4.848e{-}12$	$3.694e{-}11$	$5.350e{-}13$
R (m)	6.371e6	6.371e6	6.371e6

The result of our original analysis is shown in Figure 1. For this publication, the result is plotted in Cartesian coordinates to capture change over time. Some jaggedness is induced due to the truncation of the infinite sum and was reduced using a simple gaussian smoothing function (for graph clarity). The Gaussian smoothing does reduce the overall amplitude of the wave, and would have to be accounted for if the initial amplitude was physically based, which is not currently the case. See the code in the appendix for a time based version of the wave's coordinates transformed to spherical coordinates.

Importantly, the solution matches the stated boundary conditions and expected behavior. The solution is periodic over 2π as the boundary conditions require. Initially, the entire surface is at rest, except for the initial displacement at $\theta = 0$. This displacement then travels across the surface, decreasing in amplitude due to the damping. At $\theta = \pi$, the waves traveling across the sphere's surface converge on the other side of the sphere, briefly amplifying the wave before returning back across the sphere.

3.1 Parameter Study

In addition to observing the nominal result, the impact of the γ variable was investigated through a parameter study. γ values were varied to match equiavelent damping of several geomaterials as seen in Table 1. Sandstone being an example with higher damping than Earth's crust and soda-lime being an example of decreased damping. Note that wavespeed, μ, was held constant, though this would also vary in reality.

The results of these additional simulations are shown in Figures 2 and 3. Again, the simulation results matched expectations. With the increased damping of sandstone, the wave begins to travel across the surface, but is effectively attenuated before completing a full cycle. Alternatively, the soda-lime glass with decreased damping continues to reverberate strongly, only lightly diminished after a full cycle.

4 Discussion

One major problem with this model is the uncertainty in the parameters used. Measuring the equivalent wavespeed of Earth's crust and its effective damping is difficult. Estimating the displacement caused by an asteroid is also quite complicated. Unfortunately, the model is very sensitive to these parameters.

In addition, the assumptions made to arrive at our initial equation are significant. Obviously, the composition of the earth's crust is not uniform, implying that wavespeed and damping are actually functions of position. This would disrupt the symmetry shown in these simulations, negating the superposition effects at $\theta = \pi, 2\pi$. In addition, most forms of damping are not linear nor is the earth perfectly spherical. In some ways, even the circular transform is problematic when applied to spherical objects. In reality, as the seismic wave travels from the origin to $\theta = \frac{\pi}{2}$, the affected area increases, suggesting that the wave should naturally diminish. Instead, the circular transform models the behavior as a circular string, neglecting this change in affected area.

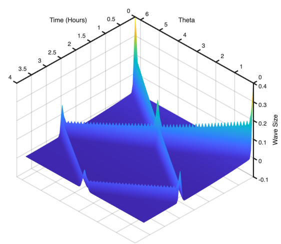

Figure 1: Simulation results of the deformation of the Earth's crust. The theta axis is the angle relative to the impact location in radians.

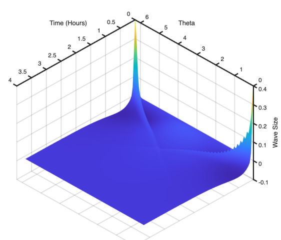

Figure 2: Simulation results for an Earth made of Sandstone. The theta axis is the angle relative to the impact location in radians.

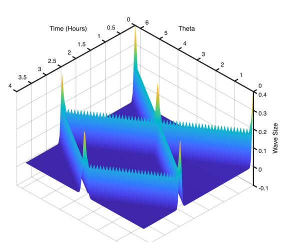

Figure 3: Simulation results for an Earth made of Soda-Lime Glass. The theta axis is the angle relative to the impact location in radians.

Certainly, a completely realistic simulation of the seismic consequences of celestial collisions is well beyond the scope of this paper. Nevertheless, here a a simple, analytical model of a wave traveling across damped spherical surface is proposed that could be applied to these earth-shattering problems. While imperfect, this model still demonstrates basic dynamics of seismic events, like the importance of soil types, and can be used as a starting point or reference for future investigations.

References

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5 Appendix: Matlab Figure and Video Generation Code

```
clear; clc; close all;

%% User settings

%video export
videoFilename = 'asteroid_impact'; %
    Name of the file (no extension
    needed here)
fps = 30;                          %
    Frames Per Second
videoTime = 3;                     %
    Number of seconds for video.
makeVideo = false;                 %
    Toggle video generation for model.
saveplot = true;

% realistic parameters
R = 6371000; %radius of sphere

mu = 4500;  %reyleigh wave velocity in
     crust, "Jeffreys-Bullen A" earth
    model. m/s

%derrive damping from Seismic Quality
    Factor (100-300 for upper mantle/
    surface waves)
Q = 21; %160: Anderson and Kovach
    results from Knopoff. %21:
    sandstone, 1450: silica, which is
    a large portion of the earths
    crust. 1340: soda lime glass
freq = 1/200; % wave frequency, as a
    function of reighley wave period,
    Knopoff. (75-300 seconds, Ben-
    Menahem, as mentioned by Knopoff).
Gamma = pi*freq/(Q*mu^2); %derrived
    from Knopoff

tspan = [0, 3*pi*R/mu]; %1.5 cycles
    with low damping.
timestep = tspan(2)/(fps*videoTime);
nTerms = 50; %number of terms in
    approximation
S = .1; %dirac delta scalar term.
NumPtsTheta = 360*2; %number of points
     around circle to calculate

% % toy parameters for testing:
% R = 1; %radius of sphere
% Gamma = 0.05; %damping is small so
    the system will be underdamped,
    creating oscillating waves.
% mu = 3;  %wave velocity, speed of
    sound in rock.
% tspan = [0,3];
% timestep = 1/fps;
% nTerms = 50; %number of terms in
    approximation
% S = 1; %dirac delta scalar term.
% NumPtsTheta = 360; %number of points
     around circle to calculate

%analysis variables
ts = (tspan(1):timestep:tspan(2)); %in
     hours
th = linspace(0,2*pi,NumPtsTheta);

%truncation adjustment parameter
sigma = 0.1; % apply smoothing to
    reduce artifacts from Gibbs
    phenomena. This effectively widens
     the impulse from a point to an
    area.


%infinite series solution:
sol = zeros(length(th), length(ts));
for i = 1:length(ts)
    t = ts(i);
    for n = 0:nTerms
        %Initial displacement method
        if (mu^4*Gamma^2 > n*mu^2/R^2)
            %real solution in laplace
               domain
            %roots of denominator
            r = [
                -Gamma*mu^2 + sqrt(
                   Gamma^2*mu^4 - (n
                   ^2*mu^2/R^2));
```



```
                -Gamma*mu^2 - sqrt(
                    Gamma^2*mu^4 - (n
                    ^2*mu^2/R^2));
            ];
            d = 2*Gamma*mu^2;
            u_n = S*cos(n*th)./(r(2)-r
                (1))*((d-r(1))*exp(r
                (1)*t) - (d-r(2))*exp(
                r(2)*t));
        else
            %complex solution from
                completing squares (
                forcing equation
            %into sine and cosine
                laplace versions for
                inverse transformation
                .
            a = Gamma*mu^2;
            b = sqrt((n^2*mu^2/R^2) -
                (Gamma*mu^2)^2);
            u_n = S*cos(n*th) * ( exp
                (-a*t)*cos(b*t) + ((a/
                b)*exp(-a*t)*sin(b*t))
                 );
        end

        %gausian filter to reduce
            gibbs phenomena from
            solution
        smoothing_factor = exp(-0.5 *
            (n * sigma)^2);
        u_n = u_n * smoothing_factor;

        %account for different n = 0
            term.
        if n == 0
            sol(:,i) = sol(:,i) +
                (1/2/pi)*(u_n)';
        else
            sol(:,i) = sol(:,i) + (1/
                pi)*(u_n)';
        end
    end
end

f = figure;
[X, Y] = meshgrid(th, ts/3600);
surf(X,Y,sol')
xlabel('Theta')
xlim([0, 2*pi])
ylabel('Time (Hours)')
zlabel('Wave Size')
view(45,135)
shading interp;
theme(f,'light')
set(gca,'color','white')
f.Color = 'w'; %Set background color
    of figure window
ax = gca;
ax.LineWidth = 2;
ax.GridLineWidth = 1.5;

if saveplot
    saveas(f,['AsteroidImpactTimeQ_'
        num2str(Q)],'fig')
    saveas(f,['AsteroidImpactTimeQ_'
        num2str(Q)],'png')
end

if makeVideo
    pause;
    close(f);
else
    return;
end



%% create video (from AI)
% code from gemini to change
    coordinates and make video.
% --- 6. Visualization Scaling ---
max_wave = max(max(abs(sol)));
if max_wave == 0, max_wave = 1; end
visual_scale_factor = (0.2 * R) /
    max_wave;

fprintf('Step 2/3: Setting up Video
    Writer...\n');

f = figure('Color','white', 'Position'
    , [100 100 800 600], 'Visible', '
```



```
    off');
theme(f, "light");

% Setup Sphere Mesh
n_res = 60;
[Phi, Theta] = meshgrid(linspace(0, pi
    , n_res), linspace(0, 2*pi, n_res)
    );

% Initialize Base Sphere
X_base = R .* sin(Phi) .* cos(Theta);
Y_base = R .* sin(Phi) .* sin(Theta);
Z_base = R .* cos(Phi);
h = surf(X_base, Y_base, Z_base);

% Styling
axis equal;
limit = R * 1.5;
axis([-limit limit -limit limit -limit
     limit]);
grid off; box on;
xlabel('X'); ylabel('Y'); zlabel('Z');
shading interp;
lighting gouraud;
light('Position',[1 1 1]);
colormap jet;
view(3);

% --- 8. Create Video Writer Object
    ---
v = VideoWriter(videoFilename, 'MPEG-4
    ');
v.FrameRate = fps;
open(v);

% --- 9. Render and Write Frames ---
fprintf('Step 3/3: Rendering Frames
    (0%%)   ');
num_frames = length(ts);

for i = 1:num_frames

    % Physics Mapping
    current_wave = sol(:, i);
    displacement = interp1(th,
        current_wave, Phi, 'linear',
        0);
    R_dynamic = R + (displacement *
        visual_scale_factor);

    X = real(R_dynamic .* sin(Phi) .*
        cos(Theta));
    Y = real(R_dynamic .* sin(Phi) .*
        sin(Theta));
    Z = real(R_dynamic .* cos(Phi));

    % Update Figure Data
    set(h, 'XData', X, 'YData', Y, '
        ZData', Z, 'CData',
        displacement);
    title(['Asteroid Impact: t = '
        num2str(ts(i)/3600, '%.2f') '
        Hours']);

    % Capture the frame
    frame = getframe(f);
    writeVideo(v, frame);

    % Simple progress bar
    if mod(i, 20) == 0
        fprintf('\b\b\b\b\b%3.0f%%)',
            (i/num_frames)*100);
    end
end

close(v); % Finish the video file
close(f); % Close the hidden figure
fprintf('\nDone! Video saved as "%s.
    mp4"\n', videoFilename);
```

