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ANALYSIS OF IMPACT GEOMETRY ON WAVE PROPAGATION IN A 2D CIRCULAR DOMAIN

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ABSTRACT

Previous 1-D stone-skipping models neglect radial energy decay and source geometry. This study models a 2-D circular domain using the Polar Wave Equation. Fourier-Bessel series solutions are derived to analyze two Gaussian initial conditions: a localized "Pebble" ($\sigma^2 = 0.05$) and a distributed "Boulder" ($\sigma^2 = 5.0$). The localized impact generated a high-frequency dispersive ripple, whereas the distributed impact acted as a geometric low-pass filter, producing a coherent, non-dispersive swell. Results demonstrate that impactor size dictates the wavefront's spectral composition and that 2-D geometry is required to capture physical amplitude attenuation.

NOMENCLATURE

$u(r, t)$: Vertical surface displacement [m]

r, θ: Polar spatial coordinates [m, rad]

R: Radius of the fluid domain (10 m)

c: Wave propagation speed (1 m/s)

σ^2: Spatial variance of the impactor [m^2]

J_0: Bessel function of the first kind, order zero

λ_n: Eigenvalues satisfying boundary conditions

A_n: Fourier-Bessel series coefficients

INTRODUCTION

In a previous paper, Nyborg (2023) modeled a stone skipping across a pond to explore wave interactions with different boundary conditions. This was done by modeling the surface of the pond with a one-dimensional wave equation and using Dirac Delta Functions to simulate the stone impacting the water at different points. Nyborg's model successfully showed how boundary conditions impact wave reflection.

A drawback of the one-dimensional model is that it does not account for radial energy decay and assumes waves maintain constant amplitude as they travel. Additionally, using the Dirac Delta Function to simulate the stone models the stone as a point source. In physical stone skipping scenarios, the "sharpness" of the impact plays an important role in determining the spectral content of the wave formation. The purpose of this paper is to explore the effect of impact sharpness or "stone size" on the resulting wavefront by modeling two different sized stones, approximated as Gaussian distributions.

METHODS

To model the vertical displacement the two-dimensional Polar Wave Equation is used. To isolate the effect of the impact geometry on the resulting wave, it is assumed that the system is axisymmetric. Physically this would represent a stone being dropped in the middle of a pond, rather than a stone being skipped across the surface. This assumption results in equation 1.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \qquad (1)$$

Boundary Conditions
1 u(R, t) = 0 (fixed rim)
2 |u(0, t)| < ∞ (finite center)

Similar to Nyborg's model, Dirichlet conditions are applied at the perimeter R=10. The second needed boundary condition is that the center must be finite to simulate a real pond.

Solution Strategy

Assuming the solution can be factored into two independent spatial and time components results in equation (2).

$$u(r,t) = \phi(r)h(t) \quad (2)$$

Using the separation of variables and substituting equation 2 into equation 1 results in two ordinary differential equations that act as Sturm-Liouville problems linked by the separation constant$-\lambda^2$.

$$\text{Time: } h''^{(t)} + c^2\,\lambda^2\,h(t) = 0 \quad (3)$$

$$\text{Space: } \phi''(r) + \frac{1}{r}\phi'(r) + \lambda^2\phi(r) = 0 \quad (4)$$

Equation (4) is Bessel's Differential Equation of order 0. The known general solution to this is $\phi(r) = C_1J_0(\lambda r) + C_2Y_0(\lambda r)$ and by applying the finiteness boundary condition it is found that $C_2 = 0$ because $Y_0 \to -\infty$. The $u(R,t) = 0$ condition means $J_0(\lambda R) = 0$ which is used to find discrete eigenvalues λ_n.

Equation (3) is a simple harmonic oscillator. $t = 0$ is defined as the moment of maximum cavity formation made by the rock where the fluid is momentarily stationary before rebounding ($u_t(r,0) = 0$). This means the sine term disappears resulting in $h(t) \propto \cos(\omega_n t)$.

Superimposing these solutions results in a Fourier-Bessel series solution for displacement u(r,t).

$$u(r,t) = \sum_{n=1}^{\infty} A_nJ_0(\lambda_n r)\cos(\omega_n t) \quad (5)$$

Initial Conditions

The depression left by the impactor is approximated as an inverted Gaussian function (6) where σ^2 represents the effective radius of the impactor.

$$f(r) = -e^{-\frac{r^2}{\sigma^2}} \quad (6)$$

By making two cases, the effect of size can be compared

Case A (Pebble): A narrow variance ($\sigma^2 = 0.05$) approximating Nyborg's point source.

Case B (Boulder): A wide variance ($\sigma^2 = 5.0$) representing a distributed displacement.

The $A_n$coefficients were calculated numerically by projecting these Gaussian shapes into the Bessel function basis by using the orthogonality property of Bessel functions with respect to weight r.

$$A_n = \frac{\int_0^R f(r)J_0(\lambda_n r)r\,dr}{\int_0^R [J_0(\lambda_n r)]^2 r\,dr} = \frac{2}{R^2[J_1(\lambda_n R)]^2}\int_0^R f(r)J_0(\lambda_n r)r\,dr$$

RESULTS

The analytical Fourier-Bessel series solution was calculated using the first 50 Bessel modes. This was required because the high frequency wave made by the pebble requires many Bessel modes to be properly represented. The simulation had a domain of R = 10 m and a speed of c = 1m/s. The figures below show the wave 4 seconds after the cavity made by the stone would be at its maximum.

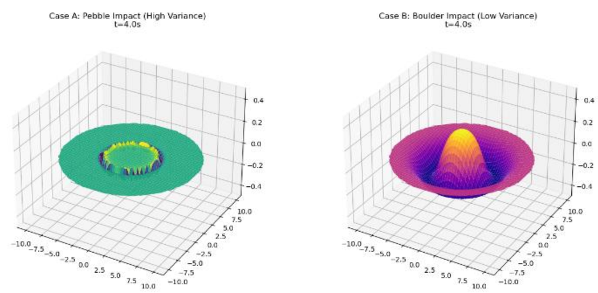

Figure 1. The 3D surface topology at t=4s. The "Pebble" (Left) exhibits a sharp, distinct ring, while the "Boulder" (Right) exhibits a single smooth swell.

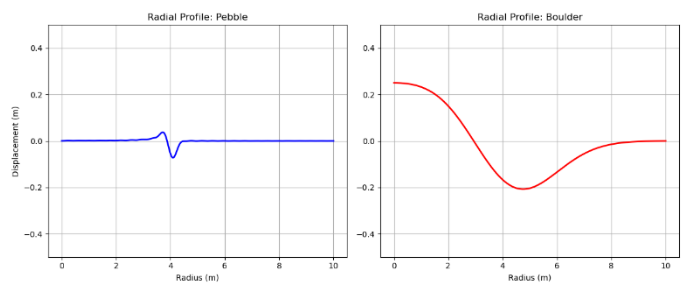

Figure 2. The radial amplitude at t=4s. When viewed over time amplitude decay is demonstrated.

It is difficult to see in these graphs but the high frequency wave made by the pebble is not as stable as large low frequency wave made by the boulder. It is easier to observe when the domain extended but the high frequency wave is rippling. This is because of the many modes that are being used to represent it.

CONCLUSIONS

The pebble simulates Nyborg's Dirac Delta Function. This high frequency wave excites high order Bessel modes and makes a noisy wave. This study focuses on the wave front but if there were multiple waves this noise would impact the wave interactions. The large boulder geometry makes a wave that is smooth and coherent because it does not excite high order Bessel modes. This confirms that the impactor's geometry acts as a spectral filter, demonstrating that impact size, not just intensity, is crucial to wave formation.

This model also shows radial energy decay. Unlike the one-dimensional model, this two-dimensional radial model demonstrates amplitude attenuation more consistent with physical energy conservation. Applying radial decay to a study like Nyborg's would help increase the realism.

However, while radial decay is present, this model still relies on simplified assumptions that impact its realism. The stone is simulated as a static initial displacement of the water. Physically, a dynamic impact involves complex cavity formation and secondary splash dynamics that generate wave trains. Additionally, the assumption of axisymmetry was used to reduce computational complexity. Future work could improve upon this by introducing angular dependence to simulate non-vertical impacts and off-center source terms to capture asymmetric reflections.

REFERENCES

[1] Nyborg, C., 2023, "Analysis of a Stone Skipping Across the Surface of Water," Journal of Applied Engineering Mathematics, 10, pp. 1-2.

[2] Solovjov, V., "Integrated Engineering Mathematics," Chapter VII, Section 6 (Bessel Functions & Fourier-Bessel Series)

APPENDIX

Python code used for simulation.

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from scipy.special import j0, j1, jn_zeros

# --- PARAMETERS ---
R = 10.0        # Radius of pond
c = 1.0         # Wave speed
N_modes = 50    # Number of Bessel modes
time_to_plot = 4.0 # Time snapshot

# --- MATH SETUP ---
roots = jn_zeros(0, N_modes)
lambdas = roots / R
omegas = c * lambdas

# Integration function for coefficients An
def calculate_An(func_shape):
  A_n = []
  r_int = np.linspace(0, R, 1000)
  dr = r_int[1] - r_int[0]
  for n in range(N_modes):
    lam = lambdas[n]
    # Integral: f(r) * J0(lambda*r) * r
    integrand = func_shape(r_int) * j0(lam * r_int) * r_int
    integral_val = np.sum(integrand) * dr
    norm_factor = 2 / ((R**2) * (j1(lam * R)**2))
    A_n.append(norm_factor * integral_val)
  return np.array(A_n)

# --- DEFINE SHAPES (The "Dent") ---
# Negative Gaussian to simulate a crater/cavity
pebble_shape = lambda r: -1.0 * np.exp(-(r)**2 / 0.05)
boulder_shape = lambda r: -1.0 * np.exp(-(r)**2 / 5.0)

# Calculate Coefficients
An_pebble = calculate_An(pebble_shape)
An_boulder = calculate_An(boulder_shape)

# --- PLOTTING DATA GENERATION ---
# 1. Line Data (for 2D Plot)
r_line = np.linspace(0, R, 400)
u_pebble_line = np.zeros_like(r_line)
u_boulder_line = np.zeros_like(r_line)

for n in range(N_modes):
  time_part = np.cos(omegas[n] * time_to_plot)
  spatial_part = j0(lambdas[n] * r_line)
  u_pebble_line += An_pebble[n] * spatial_part *
time_part
  u_boulder_line += An_boulder[n] * spatial_part *
time_part

# 2. Surface Data (for 3D Plot)
# Create Cartesian grid
x = np.linspace(-R, R, 100)
y = np.linspace(-R, R, 100)
X, Y = np.meshgrid(x, y)
R_grid = np.sqrt(X**2 + Y**2)
R_grid[R_grid > R] = np.nan # Mask outside

Z_pebble = np.zeros_like(R_grid)
Z_boulder = np.zeros_like(R_grid)

# Sum modes for 3D
for n in range(N_modes):
  time_part = np.cos(omegas[n] * time_to_plot)
  # Use J0 on the 2D grid
  # We use numpy.nan_to_num to handle the masked NaNs
safely during calc
  spatial_part = j0(lambdas[n] * np.nan_to_num(R_grid))
  Z_pebble += An_pebble[n] * spatial_part * time_part
  Z_boulder += An_boulder[n] * spatial_part * time_part

# Restore NaNs for plotting transparency
Z_pebble[np.isnan(R_grid)] = np.nan
Z_boulder[np.isnan(R_grid)] = np.nan

# --- FIGURE 1: 3D SURFACE COMPARISON ---
fig1 = plt.figure(figsize=(14, 6))

ax1 = fig1.add_subplot(121, projection='3d')
ax1.plot_surface(X, Y, Z_pebble, cmap='viridis',
edgecolor='none')
ax1.set_title(f'Case A: Pebble Impact (High
Variance)\nt={time_to_plot}s')
ax1.set_zlim(-0.5, 0.5)

ax2 = fig1.add_subplot(122, projection='3d')
```



```
    ax2.plot_surface(X, Y, Z_boulder, cmap='plasma',
edgecolor='none')
    ax2.set_title(f'Case B: Boulder Impact (Low
Variance)\nt={time_to_plot}s')
    ax2.set_zlim(-0.5, 0.5)

    plt.tight_layout()
    plt.show()

    # --- FIGURE 2: 2D RADIAL PROFILE ---
    fig2, (ax3, ax4) = plt.subplots(1, 2, figsize=(12, 5))

    ax3.plot(r_line, u_pebble_line, 'b-', lw=2)
    ax3.set_title("Radial Profile: Pebble")
    ax3.set_xlabel("Radius (m)")
    ax3.set_ylabel("Displacement (m)")
    ax3.grid(True)
    ax3.set_ylim(-0.5, 0.5)

    ax4.plot(r_line, u_boulder_line, 'r-', lw=2)
    ax4.set_title("Radial Profile: Boulder")
    ax4.set_xlabel("Radius (m)")
    ax4.grid(True)
    ax4.set_ylim(-0.5, 0.5)

    plt.tight_layout()
plt.show()
```

Figure 1. Profiles of temperature and gas mole fraction.