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Evolution of a Two-Layer Upwelling Current

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Abstract

Upwelling currents are steady wind-driven ocean currents that occur near the coasts. Their movement, although slow, is vital to local ocean dynamics and circulation. Here I use first-principles to study the transient behavior of a slow upwelling current under a no-slip, constant shear configuration, representing the interaction of such a current with the ocean floor and a strong pycnocline.

This paper shows that the vertical location of the pycnocline is a dominant factor in the time-evolution of upwelling currents and dictates the point at which the boundary layer passes from a state independent of the shear rate to one highly sensitive to it. In this second regime, behavior is affected by the magnitude and direction of the shear, and the form of its evolution, both in shape and time, varies strongly.

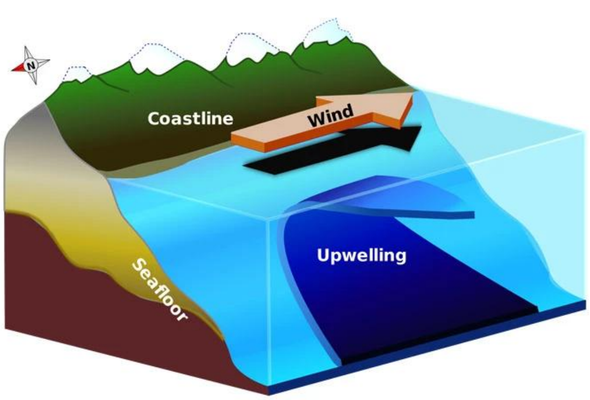

Figure 1: Wind-driven upwelling near the coast.[1]
Credit: NOAA

Nomenclature

u_i = Fluid Velocity (m/s)

τ = Shear (Pa)

μ = Dynamic Viscosity (Pa·s)

v = Kinematic Viscosity (m/s)

ρ = Fluid Density (kg/m^3)

U_0 = Current Velocity (m/s)

h = Pycnocline Height (m)

t = Time (s)

a = Shear Rate τ / μ (s^{-1})

Introduction

The coastal ocean is perhaps one of the most fascinating and relevant topics in oceanography with relation to human and marine life. Steady, dominant drivers of ocean dynamics such as the tides, surface waves and inertial forces can and often do give way to smaller, more space and time-sensitive processes such as wind-driven upwelling, internal tides or the presence of a pycnocline. Although the effects produced by these processes are often small, it is important to thoroughly understand them as they can have disproportionate impacts on their surroundings.

In this paper I investigate the time-dependance of a horizontal upwelling current through a decoupling strategy. Using the Navier-Stokes equations, I produce a simple analytical model that can help generalize trends that occur during the dual-attenuation of a transient upwelling current due to interaction with the ocean floor and an upper pycnocline (see figure 2). I then use the model to investigate how the velocities near the ocean floor vary with shear and pycnocline elevation in time.

Methodology

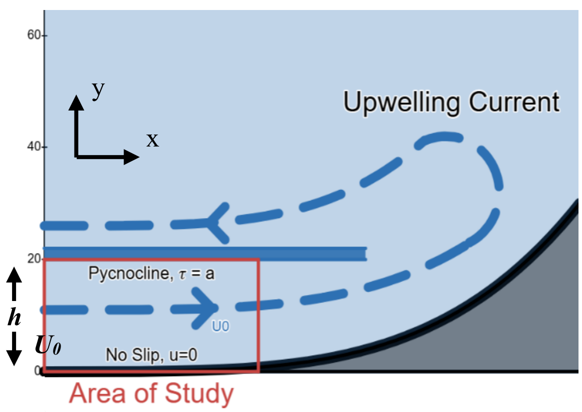

Figure 2: Setup and Coordinate System

A rectangular box of height h represents the area of study. Initially, a current U_0 travels in the positive horizontal direction and passes through the front of the box. It is initially constant in y, and downstream effects are considered negligible. In this case, τ represents steady forcing on the upper wall of the current U_0 from the bottom surface of the pycnocline. Physically, this could be produced by upwelling currents above it travelling in the opposite direction, or by other independent sources of forcing.

For this study, I decouple the upwelling problem from other processes and focus on u_i at x=0, effectively making it one-dimensional: $u_i=u$. Then I assume typical Navier-Stokes conditions and the following:

1) Infinite in x, z
2) Fully Developed Flow
3) No Applied Pressure Gradient
4) Neglecting Gravity
5) (From Continuity) dv/dy=0, v=0
6) Constant τ During Timeframe
7) Viscous Diffusion Has Minimal Effect On Flow

This reduces the Navier-Stokes equation in x to:

$$\rho\left(\frac{\partial u}{\partial t}\right) = \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

This can be split into steady-state and transient components. I first solve the steady-state problem, using the following boundary conditions (BC) for u(y, t), where: $\frac{\partial^2 u}{\partial y^2} = 0$

1) Initial Condition : $u(y, 0) = 0$
2) Bottom BC (Dirichlet) : $u(0,t) = 0$
3) Top BC: $\frac{\partial u}{\partial y}(h,t) = a$

Separating and integrating reduces the steady state problem to:

$$u(y) = ay = u_s \tag{2}$$

Where u_s is the steady state component of velocity. To solve the transient problem, I use separation of variables and Dirichlet/Neumann conditions for h, which yields the following:

$$\begin{gathered}\frac{T'}{T} = \upsilon\frac{Y''}{Y} = -\lambda^2 \\ Y'' - \lambda^2 Y = 0 \\ T' - \nu\lambda^2 T = 0\end{gathered} \tag{3}$$

The eigenfunctions $Y_n(y)$, $T_n(t)$ and eigenvalues λ can be easily shown to be:

$$\begin{gathered}Y_n(y) = \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{h}\right) \text{ for n=0,1,2…} \\ T_n(t) = e^{-\upsilon\lambda_n^2 t} \text{ for n=0,1,2…} \\ \lambda = \left(n+\frac{1}{2}\right)\frac{\pi}{h}\end{gathered} \tag{4}$$

Combining the steady-state and transient solutions I find the final solution u(y,t):

$$u(y,t) = \sum_{n=0}^{\infty} B_n \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{h}\right) e^{-\upsilon\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{h}\right)^2 t} \tag{5}$$

Where:

$$B_n = \frac{2}{h}\int_0^h (U_0 - u_s)\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{h}\right)dy$$

Variable Determination

Values for the model are carefully chosen to enable a clear picture of the area of interest.

Table 1: Values used in model

Variable	Value
h	15-20m
U_0	1 mm/s
v	0.01 m^2/s
a	$-4 \cdot 10^{-4}$ - $4 \cdot 10^{-4}$ s^{-1}

The height of an upwelling current varies based on conditions and location. In this case, the range is kept within the same order of magnitude as that found in observational research [2]. Further consideration was given to this range as being a common depth for coral reefs and other structures vital to marine life.

The value for a was selected based on the magnitude of U_0, with the upper-bound for upwelling velocities being in the ~1 mm/s [3]. a is one order of magnitude smaller, representing either relatively weak forcing or a large pycnocline.

Finally, ν was selected using the upper bound on turbulent eddy diffusion. [4] In using this value it should be acknowledged that actual timescales in the ocean would differ widely, however this assumption is considered acceptable due to the focus on the decoupled problem even if these processes would never truly function independently.

Results

The model is first run with a=0 s^{-1}, h=20 m which yields a familiar profile across time and space. This profile shows initial attenuation by the bottom surface, but not by the top, as expected:

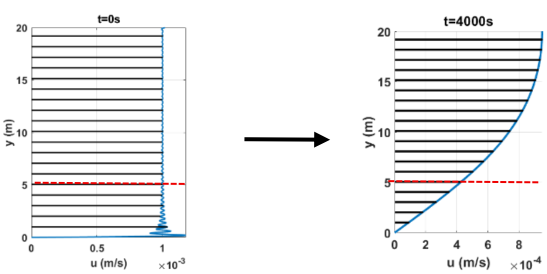

Figure 3: Evolution of boundary layer with zero top shear at t=0 and t=4000s. The red line is the focus height y=5m.

I then move on to the case of positive τ, which represents a current flowing above the pycnocline in the same direction as U_0.

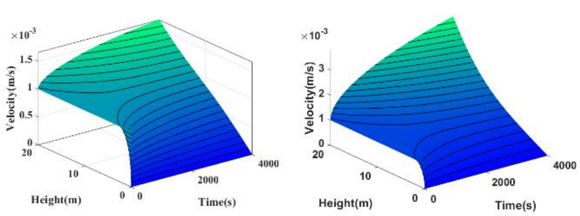

Figure 4: Boundary layer evolution for a=2·10^{-4} s^{-1} (left) and a=4·10^{-4} s^{-1} (right). Note the axis change on the right.

In both cases the profiles appear similar in shape. Dual attenuation occurs near the top and bottom surfaces, simultaneously, with rapid deterioration near the bottom surface, and a slower, logarithmic increase in velocity near the top. Both processes are effectively independent. While the bottom portion of the boundary layer can quickly reach a linear state and constant slope, as defined by the rate of diffusion, it is the slope of the shear that dictates the overall time required to achieve linear behavior. If there is mismatch in slope, the effects of the shear must diffuse all the way to the bottom to achieve a straight line, lengthening the process considerably. (see videos)

I now move to the case of negative shear, which can represent a slower-moving current in the same direction or upper upwelling currents moving in the opposite direction. It is expected that the profile for both is the same.

Without pressure driven flow, the dominant upper current must completely reverse U_0 through forcing to achieve equilibrium in u. While physical flows like this are likely rare, it is still interesting to see that there are defined areas, decreasing with height (in these cases), that retain the bulk of the velocity, which are noted on the graph. For ease of visualization the view is shifted.

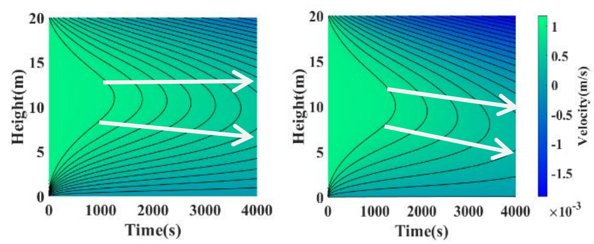

Figure 5: Boundary layer evolution for a=-2·10^{-4} s^{-1} (left) and a=-4·10^{-4} s^{-1} (right). The velocity axis applies to both. Grey arrows show evolution of bulk velocity.

While these results are interesting, they are perhaps not particularly surprising. I now combine them and look specifically at the evolution at y=5m.

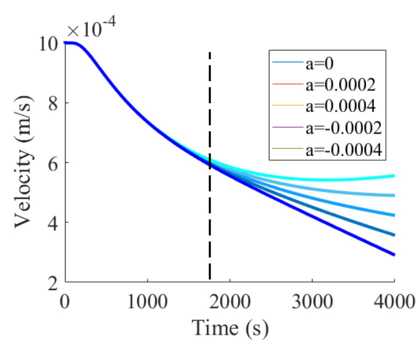

Figure 6: Variation of u_i (m/s) at y=5m with pycnocline height 20m for various shear rates *a* (1/s)

Here it can be shown that u_i remains unchanged until about ~1600s, irrespective of shear rate, a trend which remains consistent even at shear rates an order of magnitude larger. In this case the explanation is simple, shear diffusion at this height takes ~1600s to reach 5m. Doing the same at 15 m, one finds the following:

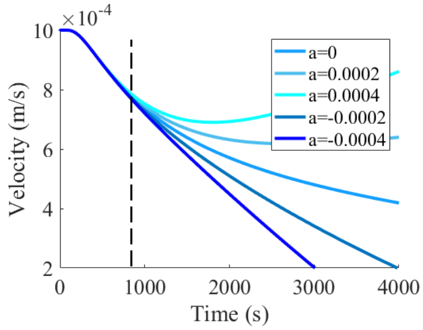

Figure 7: Variation of u_i (m/s) at y=5m with pycnocline height 15m for various shear rates *a* (1/s)

In this case diffusion is significantly more rapid and so is shear rate divergence.

Conclusions

This study has shown that a relatively strong upwelling current U_0 bounded by the ocean floor and a pycnocline is sensitive to both the location of the pycnocline and the strength and direction of the upper shearing force. At locations near the ocean floor, the evolution of U_0 can be split into two regimes, with the first being independent of the shear rate *a*, and the second being highly sensitive to it.

For pycnoclines elevations far above the floor, under more realistic conditions (slower U_0, less diffusion etc), timescales for shear diffusion to the lower layer are likely exceptional even under large shear rates. With such weak currents it is unlikely that *u*(y,t) would remain free of outside influences under these timescales. Other processes such as tidal forces, winds and currents would strongly affect upwelling far before the influence of upper forcing would diffuse to that point.

This also shows that currents below pycnoclines at lower elevations are sensitive to the rate of shear. When the shear rate slope and that of the bottom no-slip condition match, timescales are minimized and the system rapidly moves towards a linear state. Mismatch in the slopes requires more time for the profile to resemble steady-state behavior, and negative shear changes both the shape of the profile and significantly increases the time for it to appear linear.

Finally, due to the effects of diffusion, one can see that profiles in negative shear have elevations that retain much of the initial velocity for quite some time despite large changes near the top and bottom.

Acknowledgements

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References

[1] Ocean Exploration, N. (2020, September 29). *What is upwelling?*. NOAA Ocean Exploration. https://oceanexplorer.noaa.gov/ocean-fact/upwelling/

[2] Georg, T., Neves, M. C., & Relvas, P. (2023). The signature of NAO and EA climate patterns on the vertical structure of the Canary Current upwelling system. *Ocean Science*, *19*(2), 351–361. https://doi.org/10.5194/os-19-351-2023

[3] Zhang, S., Qiao, L., Gao, F., Yao, Z., & Liu, X. (2024). Intra-tidal upwelling variability off Zhoushan Islands, East China Sea. *Estuarine, Coastal and Shelf Science*. https://doi.org/10.1016/j.ecss.2024.108635.

[4] Panda, J., Sasmal, K., Maity, S., & Warrior, H. (2020). A Simple Nonlinear Eddy Viscosity Model for Geophysical Turbulent Flows. *Journal of Applied Fluid Mechanics*. https://doi.org/10.47176/jafm.14.03.32032.

Appendix

This includes the code and links to videos of the evolution of the boundary layer.

Videos:

https://drive.google.com/drive/folders/13jE5cyv3Z7O_LFMtfqXsSFrgT92uayCN?usp=drive_link

Code:

```
%% Part 1 - Calculating and Visualizing the Boundary Layer Evolution %%

clear
clf

k = VideoWriter('couette_transient14.mp4', 'MPEG-4'); % Video Setup and Title
k.FrameRate = 80;
open(k);

h=15; y=0:0.01:h; t=0:2.5:4000; % Variable definition and setup
v=0.01; u=0.001; a=0.0004; %-0.0002:0.0001:0.0002;
figure(1); clf; hold on;
h_plot = plot(0,0);
u1=zeros(length(t), length(y));

for i=1:length(t) % Calculates the boundary profile for all t
   r=0;
   t1=t(i);
   u1(i,:) = a.*y;
for n = 0:100 % Calculates the boundary profile for a given t with a large series
      r1 = -(2000 * a * h * sin((2 * pi * n + pi) / 2) + ((2 * pi - 2000 * pi * a * h) * n - 1000 * pi * a * h + pi) * cos((2 * pi * n + pi) / 2) - 2 * pi * n - pi) / (250 * pi^2 * (2 * n + 1)^2);
      r2 = sin((n+1/2)*pi*y/h);
      r3 = exp(-v*((n+1/2)*pi/h)^2*t1);
      r = r + r1.*r2.*r3;
end

   u1(i,:) = u1(i,:) + r;
   xlabel('u (m/s)'); ylabel('y (m)'); ylim([0 h]);
   %xlim([0 a*h-0.1]); % Manual Fix
   grid on; title('Transient Couette Flow')
   set(h_plot, 'XData', u1(i,:), 'YData', y);
   drawnow;
   title(sprintf('Transient Couette Flow, t = %.1f s', t1));

   frame=getframe(gcf); % Adds frame to video
   writeVideo(k, frame);

end

close(k);
hold on; save('15_4.mat', 'u1', 't'); % Saves dataset for each iteration, title needs to be changed manually
figure(1); idx = 1:101:length(y);
quiver(zeros(size(y(idx))), y(idx), u1(1601,idx), zeros(size(y(idx))), 0, ...
      'Color', 'k', 'LineWidth', 0.8, 'MaxHeadSize', 2000);
axis tight;
figure(300); hold on;
h = surf(t, y, u1'); shading interp; contour3(t, y, u1', 20, 'k', 'LineWidth', 1); view(3);
xlabel('Time(s)'); ylabel('Height(m)');
zlabel('Velocity(m/s)'); colormap('winter');


%% Part 2 - Creating evolution of u at y=5m %%
% clear; close all;
% figure(101); clf; hold on;
% for n=1:10
%     k = [0, 2, 4, -2, -4, 0, 2, 4, -2, -4,];
%     if n<6, load(sprintf('20_%i.mat', k(n))); else, load(sprintf('20_%i.mat', k(n))); end
%     u2(:, 1) = u1(:, 501); t2(1,:) = t(1,:);
%     gcf;
%     plot(t2, u2)
% end
%
% title('Variation of U (m/s) at Y=5m for Various a (1/s) Shear Values');
% legend('a=0', 'a=0.0002', 'a=0.0004', 'a=-0.0002', 'a=-0.0004');
% xlabel('Time (s)');
% ylabel('Velocity (m/s)');
```

