

## Evolution of a Pycnocline-Bounded Upwelling Current

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### Abstract

Upwelling currents are steady wind-driven ocean currents that occur near the coasts [1]. Their movement, although slow, is vital to local ocean dynamics and circulation. Here I use first principles to study the transient behavior of a slow upwelling current under a no-slip, constant shear configuration, representing the interaction of such a current with the ocean floor and a strong pycnocline (a thin, strongly stratified layer).

This paper shows that the vertical location of the pycnocline is a dominant factor in the time-evolution of upwelling currents and dictates the point at which the boundary layer passes from a state independent of the shear rate to one highly sensitive to it. In this second regime, behavior is affected by the magnitude and direction of the shear, and the form of its evolution, both in shape and time, varies strongly.

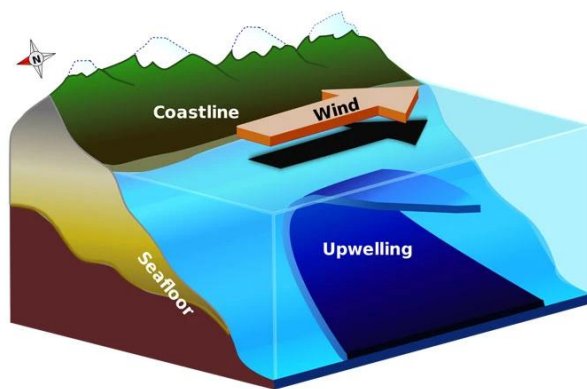


Figure 1: Wind-driven upwelling near the coast.[1]  
Credit: NOAA

### Nomenclature

$u_i$  = Fluid Velocity (m/s)

$\tau$  = Shear (Pa)

$\mu$  = Dynamic Viscosity (Pa·s)

$\nu$  = Kinematic Viscosity (m/s)

$\rho$  = Fluid Density (kg/m<sup>3</sup>)

$U_0$  = Current Velocity (m/s)

$h$  = Pycnocline Height (m)

$t$  = Time (s)

$a$  = Shear Rate  $\tau / \mu$  (s<sup>-1</sup>)

### Introduction

The coastal ocean is perhaps one of the most fascinating and relevant topics in oceanography with relation to human and marine life. Steady, dominant drivers of ocean dynamics such as the tides, surface waves and inertial forces can and often do give way to smaller, more space and time-sensitive processes such as wind-driven upwelling, internal tides or the presence of a pycnocline. Although the effects produced by these processes are often small, it is important to thoroughly understand them as they can have disproportionate impacts on their surroundings.

In this paper I investigate the time-dependance of a horizontal upwelling current through a decoupling strategy. Using the Navier-Stokes equations, I produce a simple analytical model that can help generalize trends

that occur during the dual-attenuation of a transient upwelling current due to interaction with the ocean floor and an upper pycnocline (Figure 2). I then use the model to investigate how the velocities near the ocean floor vary with shear and pycnocline elevation in time.

## Methodology

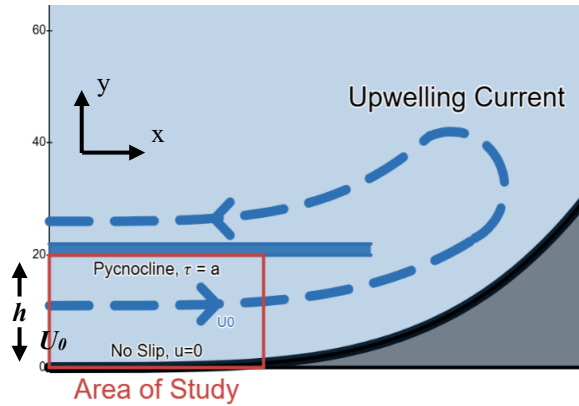


Figure 2: Setup and Coordinate System

A rectangular box of height  $h$  represents the area of study. Initially, a current  $U_0$  travels in the positive horizontal direction and passes through the front of the box. It is initially constant in  $y$ , and downstream effects are considered negligible. In this case,  $\tau$  represents steady forcing on the upper wall of the current  $U_0$  from the bottom surface of the pycnocline. Physically, this could be produced by upwelling currents above it travelling in the opposite direction, or by other independent sources of forcing.

For this study, I decouple the upwelling problem from other processes and focus on  $u$ ; at  $x=0$ , effectively making it one-dimensional:  $u_i=u$ . Then I assume typical Navier-Stokes conditions and the following:

- 1) Infinite in  $x, z$
- 2) Fully Developed Flow
- 3) No Applied Pressure Gradient
- 4) Neglecting Gravity
- 5) (From Continuity)  $dv/dy=0, v=0$
- 6) Constant  $\tau$  During Timeframe
- 7) Viscous Diffusion has Minimal Effect on Flow

This reduces the Navier-Stokes equation in  $x$  to:

$$\rho \left( \frac{\partial u}{\partial t} \right) = \mu \left( \frac{\partial^2 u}{\partial y^2} \right) \quad (1)$$

This can be split into steady-state and transient components. I first solve the steady-state problem, using the following boundary conditions (BC) for  $u(y, t)$ , where:  $\frac{\partial^2 u}{\partial y^2} = 0$

- 1) Initial Condition :  $u(y, 0) = 0$
- 2) Bottom BC (Dirichlet) :  $u(0, t) = 0$
- 3) Top BC:  $\frac{\partial u}{\partial y}(h, t) = a$

Separating and integrating reduces the steady state problem to:

$$u(y) = ay = u_s \quad (2)$$

Where  $u_s$  is the steady state component of velocity. To solve the transient problem, I use separation of variables and Dirichlet/Neumann conditions for  $h$ , which yields the following:

$$\begin{aligned} \frac{T'}{T} &= v \frac{Y''}{Y} = -\lambda^2 \\ Y'' - \lambda^2 Y &= 0 \\ T' - v\lambda^2 T &= 0 \end{aligned} \quad (3)$$

The eigenfunctions  $Y_n(y)$ ,  $T_n(t)$  and eigenvalues  $\lambda$  can be easily shown to be:

$$\begin{aligned} Y_n(y) &= \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi y}{h} \right) \text{ for } n=0,1,2,\dots \\ T_n(t) &= e^{-v\lambda_n^2 t} \text{ for } n=0,1,2,\dots \\ \lambda &= \left( n + \frac{1}{2} \right) \frac{\pi}{h} \end{aligned} \quad (4)$$

Combining the steady-state and transient solutions I find the final solution  $u(y, t)$ :

$$u(y, t) = \sum_{n=0}^{\infty} B_n \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi y}{h} \right) e^{-v \left( \left( n + \frac{1}{2} \right) \frac{\pi}{h} \right)^2 t} \quad (5)$$

Where:

$$B_n = \frac{2}{h} \int_0^h (U_0 - u_s) \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi y}{h} \right) dy$$

## Variable Determination

Values for the model are carefully chosen to enable a clear picture of the area of interest.

Table 1: Values used in model

Variable	Value
$h$	15-20m
$U_0$	1 mm/s
$v$	0.01 m <sup>2</sup> /s
$a$	$-4 \cdot 10^{-4} - 4 \cdot 10^{-4} \text{ s}^{-1}$

The height of an upwelling current varies based on conditions and location. In this case, the range is kept within the same order of magnitude as that found in observational research [2]. Further consideration was given to this range as being a common depth for coral reefs and other structures vital to marine life.

The value for  $a$  was selected based on the magnitude of  $U_0$ , with the upper-bound for upwelling velocities being in the  $\sim 1$  mm/s [3].  $a$  is one order of magnitude smaller, representing either relatively weak forcing or a large pycnocline.

Finally,  $\nu$  was selected using the upper bound on turbulent eddy diffusion. [4] In using this value it should be acknowledged that actual timescales in the ocean would differ widely. In this case, this assumption is considered acceptable due to the focus on the decoupled problem even if these processes do not function independently.

## Results

The model is first run with  $a=0$  s<sup>-1</sup>,  $h=20$  m (Figure 3) which yields a familiar profile across time and space. This profile shows initial attenuation by the bottom surface, but not by the top, as expected:

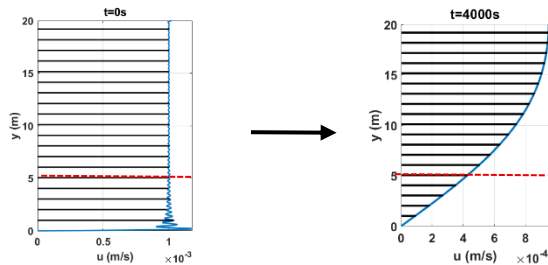


Figure 3: Evolution of boundary layer with zero top shear at  $t=0$  and  $t=4000$ s. The red line is the focus height  $y=5$ m.

I then move on to the case of positive  $\tau$  (Figure 4), which represents a current flowing above the pycnocline in the same direction as  $U_0$ .

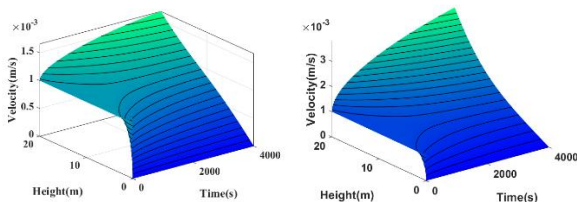


Figure 4: Boundary layer evolution for  $a=2 \cdot 10^{-4}$  s<sup>-1</sup> (left) and  $a=4 \cdot 10^{-4}$  s<sup>-1</sup> (right). Note the axis change on the right.

In both cases the profiles appear similar in shape. Dual attenuation occurs near the top and bottom surfaces, simultaneously, with rapid deterioration near the bottom surface, and a slower, logarithmic increase in velocity near the top. Both processes are effectively independent. While the bottom portion of the boundary layer can quickly reach a linear state and constant slope, as defined by the rate of diffusion, it is the slope of the shear that dictates the overall time required to achieve linear behavior. If there is mismatch in slope, the effects of the shear must diffuse all the way to the bottom to achieve a straight line, lengthening the process considerably (see video).

I now move to the case of negative shear (Figure 5), which could represent a slower-moving current in the same direction or upper upwelling currents moving in the opposite direction. The profiles for both should look the same.

Without pressure driven flow, the dominant upper current must completely reverse  $U_0$  through forcing to achieve equilibrium in  $u$ . While physical flows like this are likely rare, it is still interesting to see that there are certain regions in time that retain the bulk of the velocity, which are noted on the graph. For ease of visualization, the figures now show the top-down projection.

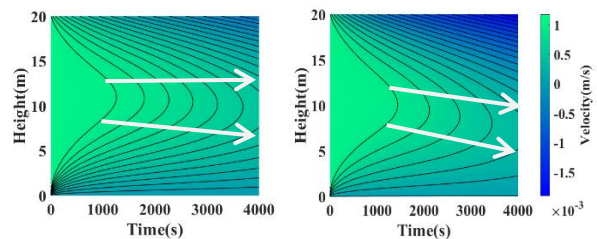


Figure 5: Boundary layer evolution for  $a=-2 \cdot 10^{-4}$  s<sup>-1</sup> (left) and  $a=-4 \cdot 10^{-4}$  s<sup>-1</sup> (right). The velocity axis applies to both. Grey arrows show evolution of bulk velocity.

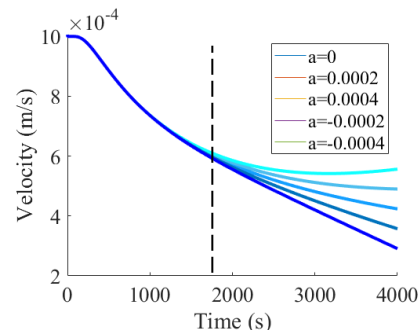


Figure 6: Variation of  $u_i$  (m/s) at  $y=5$ m with pycnocline height 20m for various shear rates  $a$  (1/s)

While these results are interesting, they are not particularly surprising. To get an idea of the trends in velocity  $u$  across time near the bottom, I combine the results and look specifically at the evolution at  $y=5\text{m}$  (Figure 6).

Here it can be shown that  $u_i$  remains unchanged until about  $\sim 1600\text{s}$ , irrespective of shear rate, a trend which remains consistent even at shear rates an order of magnitude larger. In this case it is simply because shear diffusion takes  $\sim 1600\text{s}$  to reach  $5\text{m}$  at this height. Doing the same at  $15\text{m}$  (Figure 7), one finds the following:

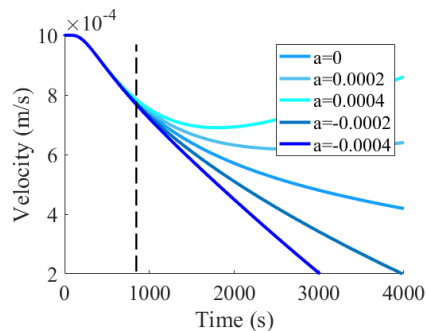


Figure 7: Variation of  $u_i$  (m/s) at  $y=5\text{m}$  with pycnocline height  $15\text{m}$  for various shear rates  $a$  (1/s)

In this case diffusion is significantly more rapid and so is shear rate divergence.

## Conclusions

This study has shown that a relatively strong upwelling current  $U_0$  bounded by the ocean floor and a pycnocline is sensitive to both the location of the pycnocline and the strength and direction of the upper shearing force. At locations near the ocean floor, the evolution of  $U_0$  can be split into two regimes, with the first being independent of the shear rate  $a$ , and the second being highly sensitive to it.

For pycnocline elevations far above the floor, under more realistic conditions (slower  $U_0$ , less diffusion etc), timescales for shear diffusion to the lower layer are likely exceptional even under large shear rates. With such small velocities it is unlikely that  $u(y,t)$  would remain free of outside influences. Other processes such as tidal forces, winds and currents would strongly affect upwelling far before the influence of upper forcing would diffuse to that point.

This also shows that currents below pycnoclines are sensitive to the rate of shear. When the shear rate slope and that of the bottom no-slip condition match, timescales

are minimized and the system rapidly moves towards a linear state. Mismatch in the slopes requires more time for the growing profile to take a shape that resembles steady-state behavior, and negative shear changes both the shape of the profile and significantly increases the time for it to appear linear.

Finally, due to the effects of diffusion, one can see that profiles in negative shear have elevations that retain much of the initial velocity for quite some time despite large changes near the top and bottom.

## Acknowledgements

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## References

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- [4] Panda, J., Sasmal, K., Maity, S., & Warrior, H. (2020). A Simple Nonlinear Eddy Viscosity Model for Geophysical Turbulent Flows. *Journal of Applied Fluid Mechanics*. <https://doi.org/10.47176/jafm.14.03.32032>.

## Appendix

This includes the code and links to videos of the evolution of the boundary layer.

### Video:

<https://youtu.be/ORRy449c-EI>

### Code:

%% Part 1 - Calculating and Visualizing the Boundary Layer Evolution %%

```
clear
clf
```

```
k = VideoWriter('Couette_TransientNegativeShear.mp4',
'MPEG-4'); % Video Setup and Title
k.FrameRate = 80;
open(k);
```

```
h=15; y=0:0.01:h; t=0:10:4000; % Variable definition
and setup
v=0.01; a=-0.0004; % -0.0002:0.0001:0.0002;
figure(1); clf; hold on;
h_plot = plot(0,0);
u1=zeros(length(t), length(y));
```

```
for i=1:length(t) % Calculates the boundary profile for
all t
```

```
    r=0;
    t1=t(i);
    u1(i,:) = a.*y;
    for n = 0:100 % Calculates the boundary profile for a
    given t with a large series
        r1 = -(2000 * a * h * sin((2 * pi * n + pi) / 2) - 2 *
pi * n - pi) / (250 * pi^2 * (2 * n + 1)^2);
        r2 = sin((n+1/2)*pi*y/h);
        r3 = exp(-v*((n+1/2)*pi/h)^2*t1);
        r = r + r1.*r2.*r3;
```

```
end
```

```
    u1(i,:) = u1(i,:) + r;
```

```
    xlabel('u (m/s)'); ylabel('y (m)'); ylim([0 h]);
    xlim([-1.5E-3 1.5E-3]); % Manual Fix
    grid on;
    set(h_plot, 'XData', u1(i,:), 'YData', y);
    drawnow;
    title(sprintf('Transient Couette Flow, Negative Shear, t
= %.1f s', t1));
```

```
    frame=getframe(gcf); % Adds frame to video
    writeVideo(k, frame);
```

```
end
```

```
close(k);
hold on; save('15_t.mat', 'u1', 't'); % Saves dataset for
each iteration, title needs to be changed manually
figure(1); idx = 1:101:length(y);
quiver(zeros(size(y(idx))), y(idx), u1(size(u1,1),idx),
zeros(size(y(idx))), 0, 'Color', 'k', 'LineWidth', 0.8,
'MaxHeadSize', 2000); axis tight;
figure(300); hold on;
h = surf(t, y, u1'); shading interp; contour3(t, y, u1', 20,
'k', 'LineWidth', 1); view(3);
xlabel('Time(s)'); ylabel('Height(m)');
zlabel('Velocity(m/s)'); colormap('winter');
```

```
%% Part 2 - Creating evolution of u at y=5m %%
```

```
% clear; close all;
% figure(101); clf; hold on;
% for n=1:10
%     k = [0, 2, 4, -2, -4, 0, 2, 4, -2, -4,];
%     if n<6, load(sprintf('20_%i.mat', k(n))); else,
load(sprintf('20_%i.mat', k(n))); end
%     u2(:, 1) = u1(:, 501); t2(1,:) = t(1,:);
%    (gcf);
%     plot(t2, u2)
% end
%
% title('Variation of U (m/s) at Y=5m for Various a (1/s)
Shear Values');
% legend('a=0', 'a=0.0002', 'a=0.0004', 'a=-0.0002', 'a=-
0.0004');
% xlabel('Time (s)');
% ylabel('Velocity (m/s)');
```