Journal of Applied Engineering Mathematics

Volume 12, December 2025

WAVE CANCELLATION ON CIRCULAR MEMBRANE: SOLVING FOR TIME-DEPENDENT BOUNDARY MOTION USING HANKEL-MODE DECOMPOSITION

Isaac Sudweeks and Nik Houtz

Mechanical Engineering Department
Brigham Young University
Provo, Utah 84602
Sudweeksisaac@gmail.com | nphoutz@gmail.com

ABSTRACT

Noise cancellation is a commonly used technology that cancels out incoming waves by producing their opposite. This idea of cancelling out waves became an inspiration for our problem. We solved the wave equation with a Dirichlet boundary condition defined as a time-dependent displacement. We then solved for this displacement so that after time T, all waves on the membrane get cancelled, and we are left with no motion. Studying the motion of the waves before time T, we concluded that the boundary displacement acts in a nonintuitive way to cancel out eigenmodes rather than the wave directly. Possible further work is also presented.

NOMENCLATURE

ν = Wave speed
r = Radial coordinate measured from the membrane center
t = Time variable
r_1 =The radial position of the boundary
T = The time at which all waves should be canceled
$u(r,t)$ = Transverse displacement at radius r and time t
$g(t)$ = The boundary condition that is used to impose control on the membrane
J_0 = The zeroth-order Bessel function.
J_1 = The first-order Bessel function
λ_n = The nth eigenvalue
$\bar{u}_n$ = Hankel-transformed displacement corresponding to eigenmode n.

INTRODUCTION

Control theory provides an interesting foundation for explaining physical phenomena, especially those defined by differential equations. In this project, we studied a disk-like membrane governed by the wave equation and explored how to design a boundary condition function that drives the boundary displacement and velocity to zero at a selected final time. This problem is very interesting on account of its inverse dichotomy; instead of fixing the boundary conditions and observing the resulting waves, we instead determine the boundary displacement that is required to cancel them.

The inspiration for this project stems from the concept of noise cancellation. Modern noise cancellation technology makes use of destructive interference: the device generates the opposite amplitude of the unwanted signal and sends it to the receiving end to cancel waves out. In practice, this requires sensing the incoming waves, discretizing them into manageable pieces, and then producing an output that sums to a zero value at the receiving end or boundary.

Our project essentially emulates this idea and applies it in a 3-dimensional cylindrical context. We establish a circular membrane, apply the radial wave equation, organize initial conditions, and then derive the necessary boundary displacements to drive the signal to zero at a desired final time.

PROBLEM DEFINITION

The wave equation is defined as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2}$$

With initial conditions defined as:

$$u(r,0) = u_0(r)$$

$$\frac{\partial}{\partial t}u(r,0) = u_1(r)$$

And a Dirichlet boundary condition defined as:

$$u(r_1, t) = g(t)$$

The objective: solve $g(t)$ such that at time T: $u(r,T) = 0$ and $u'(r,T) = 0\ \forall \mathrm{r} \in [0, \mathrm{r}_1)$

METHODOLOGY

We started by establishing the wave equation and its initial conditions. Then we applied the finite Hankel transform and deduced the Eigenvalues via the Sturm-Liouville problem setup. This resulted in a non-homogeneous ODE in modal space. Solving this ODE yielded the solution of the transformed equation. After solving for the transformed equation and performing the inverse Hankel transform, we solved for the boundary displacement such that our solution would vanish. To do that, we used a finite piecewise constant ansatz to approximate $g(t)$. Substituting this form into the equations produced by the Hankel transformation left us with a linear system, which we solved to find control coefficients that were used to define boundary motion.

HANKEL TRANSFORM

The Finite Hankel Transform is defined as[1]:

$$\bar{u_n} = \int_0^{r_1} u\,(r) J_0(\lambda_n r) r dr$$

Where λ_n are the eigenvalues defined by the Sturm-Liouville problem. Because the boundary condition is of the Dirichlet type, the eigenvalues are defined as the roots of the following equation.

$$J_0(\lambda_n \mathrm{r}_1) = 0$$

The operational property of the Discrete Hankel Transform is defined as:

$$H\left\{\frac{1}{r}\frac{\partial}{\partial t}\left(r\frac{\partial u}{\partial t}\right)\right\} = r_1\lambda_n J_1(\lambda_n r_1) g(t) - \lambda_n^2 \bar{u}_n$$

Using the operational property, the equation is transformed to:

$$\frac{\partial^2 u}{\partial t^2} + \nu^2\lambda_n\bar{u}_n = \nu^2 r_1\lambda_n J_1(\lambda_n r_1) g(t)$$

With transformed initial conditions: $\bar{u}_0$ and $\bar{u}_1$

SOLVING LINEAR ODE

α is defined as:

$$\alpha = \nu^2\lambda_n r_1 J_1(\lambda_n r_1)$$

The transformed equation can be written as:

$$\frac{\partial^2 u}{\partial t^2} + \nu^2\lambda_n\bar{u}_n = \alpha g(t)$$

ω_n is defined as:

$$\omega_n = \sqrt{\nu^2\lambda_n}$$

The solution to the homogeneous transformed ODE is:

$$\bar{\mathrm{u}}_{\mathrm{hn}} = c_1\cos\left(\sqrt{\nu^2\lambda_n}t\right) + c_2\sin\left(\sqrt{\nu^2\lambda_n}t\right)$$

By Green's theorem for linear wave operators, the solution can be written as a convolution of the Green kernel with the forcing term[2].

$$\bar{u}_{pn} = \frac{\alpha}{\omega_n}\int_0^t \sin\left(\omega_n(t-\tau)\right) g(\tau) d\tau$$

The solution is then defined as the superposition of the homogeneous solution and the particular solution:

$$\bar{u}_n = \bar{u}_{hn} + \bar{u}_{pn}$$

Using the inverse Hankel transform, the solution to the wave equation is:

$$\boxed{\sum_{n=1}^{\infty}\bar{u}_n\frac{J_0(\lambda_n r)}{\frac{r_1^2}{2}J_1(\lambda_n r_1)}}$$

IMPLEMENTING CONSTRAINTS

Plugging in T for t into the solution to the transformed ordinary differential equation and solving for the convolution integral yields:

$$\int_0^T \sin\left(\omega_n(t-\tau)\right) g(\tau) d\tau = S_n$$

Solving for the convolution integral for the time derivative of the same function results in:

$$\int_0^T \cos\left(\omega_n(t-\tau)\right) g(\tau) d\tau = C_n$$

Where S_n and C_n are defined as:

$$S_n = \frac{-\omega_n}{\alpha}\left(\bar{u}_0\cos(\omega_n T) + \bar{u}_1\sin(\omega_n T)\right)$$

$$C_n = \frac{1}{\alpha}\left(\omega_n\bar{u}_1\cos(\omega_n T) - \omega_n\bar{u}_0\sin(\omega_n T)\right)$$

A piecewise constant ansatz can be used to approximate $g(t)$ because it maps linearly onto the governing equations:

$$g(t) \approx \sum_{k=1}^{M} \gamma_k\, \phi_k(t)$$

The integrals can be quantized into the same bins as $g(t)$:

$$I_s(n,k) := \int_{t_k}^{t_{k+1}} \sin\left(\omega_n(t-\tau)\right)d\tau$$

$$I_c(n,k) := \int_{t_k}^{t_{k+1}} \cos\left(\omega_n(t-\tau)\right)d\tau$$

Substituting this into the equations:

$$S_n \approx \sum_{k=1}^{M} \gamma_k\, \phi_k I_s(n,k)$$

$$C_n \approx \sum_{k=1}^{M} \gamma_k\, \phi_k I_c(n,k)$$

To solve for the coefficients of the piecewise constant ansatz, a linear equation is defined:

$$\vec{g} = A^{-1}\vec{b}$$

Where $\vec{g}$ are the coefficients of the piecewise constant ansatz and A is defined as:

$$A = \begin{bmatrix} I_s(1,1) & I_s(1,2) & \dots & I_s(1,M) \\ I_c(1,1) & I_c(1,2) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ I_s(n,1) & I_s(n,2) & \dots & I_s(n,M) \\ I_c(n,1) & I_c(n,2) & \dots & I_c(n,M) \end{bmatrix}$$

And $\vec{b}$ is defined as:

$$\vec{b} = \begin{bmatrix} S_1 \\ C_1 \\ \vdots \\ S_n \\ C_n \end{bmatrix}$$

Solving this problem yields the solution for g such that the displacement and velocity of the membrane will go to zero at time T.

RESULTS AND DISCUSSION

We found that an attempt at making sense of the solution we derived is a rather difficult task. The problem was formulated in a way that makes it hard to connect to a real-world scenario. The boundary displacement does not impose a rigid vertical motion on the membrane. Instead, it enters only through the wave equation's boundary condition, acting as a source of radial waves that propagate into the interior. The membrane responds dynamically to the boundary rather than being kinematically lifted by it. These inward propagating waves inject momentum and displacement, which interact to cancel out initial conditions.

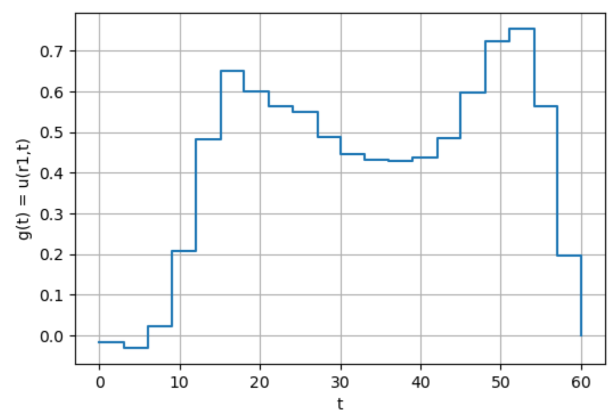

Figure 1: Boundary control of the wave equation with an initial Gaussian pulse.

As can be seen in Figure 1, g(t) seems to be unique depending on initial conditions and parameters of the membrane. As can be seen in Figure two, the movement of the wave is not constrained to any value until time T, where it goes to zero. This shows that the boundary is not necessarily simply reducing the wave over time until it reaches zero, but rather that it is exciting waves in such a way that they will interact to self-destruct at time T, which is nonintuitive.

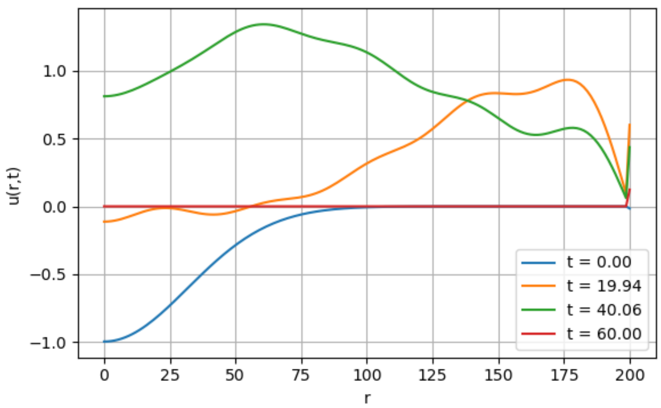

Figure 2: Membrane cross-section at various time steps

We thought it was interesting that before time T, there seemed to be no rhyme or reason to the input of the moving boundary. As discussed above, one would intuitively think that the forcing term would slowly destroy the wave over time until it became zero, but as can be seen in Figure 2, the wave many times grows before it ultimately gets cancelled out in the end.

The reason for this is that the boundary displacement is effectively wiping out one eigenmode at a time until it destroys the whole wave. These eigenmodes do not necessarily correlate with making the waves smaller, and so when the modes are wiped out, they can change the wave in unexpected ways, which leads to the erratic behavior we see.

CONCLUSIONS

Our project was an attempt at applying active noise cancellation through boundary control to a radially propagating wave on a circular membrane. After establishing the wave equation and its initial conditions, we applied the Hankel transform and the appropriate Sturm-Liouville eigenfunction problem in order to express our solution in terms of the Bessel equation, noting its eigenmodes. We then applied the inverse transform, which led us to utilize a piecewise-constant function for the boundary control, g(t). This led to the production of a linear system whose solutions cancel the outgoing waves. This approach shows how integral transform methods and constraint-based control techniques can provide an effective solution for boundary control of radial wave propagation.

Future work could include solving for a forcing term to achieve similar results. Similarly, using a more physically realistic boundary condition, such as the Robin conditions, would lead to more physically representative results.

ACKNOWLEDGMENTS

We want to acknowledge the help and guidance offered to us by Dr. Soloviev. He has been an excellent professor, having taught both Math 303 (Mathematics for Engineering 2) and ME 505 (Applied Engineering Math) to us. He has always been an incredible resource for expanding our knowledge and learning. His passion for math inspires the hearts and minds of young people everywhere

REFERENCES

[1] Soloviev, V., "INTEGRATED ENGINEERING MATHEMATICS."
[2] Strauss, W. A., 2008, *Partial Differential Equations: An Introduction*, Wiley, Hoboken, NJ.

