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WAVE CANCELLATION ON CIRCULAR MEMBRANE: SOLVING FOR TIME-DEPENDENT BOUNDARY MOTION USING HANKEL-MODE DECOMPOSITION

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ABSTRACT

Noise cancellation is a commonly used technology that cancels out incoming waves by producing their opposite. This idea of cancelling out waves became an inspiration for our problem. We solved the wave equation with a Dirichlet boundary condition defined as a time-dependent displacement. We then solved for this displacement so that after time T, all waves on the membrane get cancelled, and we are left with no motion. Studying the motion of the waves before time T, we concluded that the boundary displacement acts in an intuitive way to cancel out eigenmodes by essentially adjusting to compensate for and then destroying the incoming wave displacements over time. Possible further work is also presented.

NOMENCLATURE

v = Wave speed
r = Radial coordinate measured from the membrane center
t = Time variable
r_1 =The radial position of the boundary
T = The time at which all waves should be canceled
$u(r,t)$ = Transverse displacement at radius r and time t
$g(t)$ = The boundary condition that is used to impose control on the membrane
J_0 = The zeroth-order Bessel function.
J_1 = The first-order Bessel function
λ_n = The nth eigenvalue
$\bar{u}_n$ = Hankel-transformed displacement corresponding to eigenmode n.

INTRODUCTION

Control theory provides an interesting foundation for explaining physical phenomena, especially those defined by differential equations. In this project, we studied a disk-like membrane governed by the wave equation and explored how to design a boundary condition function that drives the boundary displacement and velocity to zero at a selected final time. This problem is very interesting on account of its inverse dichotomy; instead of fixing the boundary conditions and observing the resulting waves, we instead determine the boundary displacement that is required to cancel them.

The inspiration for this project stems from the concept of noise cancellation. Modern noise cancellation technology makes use of destructive interference: the device generates the opposite amplitude of the unwanted signal and sends it to the receiving end to cancel waves out. In practice, this requires sensing the incoming waves, discretizing them into manageable pieces, and then producing an output that sums to a zero value at the receiving end or boundary.

Our project essentially emulates this idea and applies it in a 3-dimensional cylindrical context. We establish a circular membrane, apply the radial wave equation, organize initial conditions, and then derive the necessary boundary displacements to drive the signal to zero at a desired final time.

PROBLEM DEFINITION

The wave equation is defined as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2}$$

With initial conditions defined as:

$$u(r,0)=u_0(r)$$

$$\frac{\partial}{\partial t}u(r,0)=u_1(r)$$

And a Dirichlet boundary condition defined as:

$$u(r_1,t)=g(t)$$

The objective: solve $g(t)$ such that at time T: $u(r,T)=0$ and $u'(r,T)=0\ \forall r\in[0,\mathrm{r}_1)$

METHODOLOGY

We started by establishing the wave equation and its initial conditions. Then we applied the finite Hankel transform and deduced the Eigenvalues via the Sturm-Liouville problem setup. This resulted in a non-homogeneous ODE in modal space. Solving this ODE yielded the solution of the transformed equation. After solving for the transformed equation and performing the inverse Hankel transform, we solved for the boundary displacement such that our solution would vanish. To do that, we used a finite piecewise constant ansatz to approximate $g(t)$. Substituting this form into the equations produced by the Hankel transformation left us with a linear system, which we solved to find control coefficients that were used to define boundary motion. It is helpful to note that the derivation for the Hankel Transform and other related equations will not be shown; for more information on these topics, refer to Integrated Math For Engineers [1].

HANKEL TRANSFORM

The Finite Hankel Transform is defined as [1]:

$$\bar{u_n}=\int_0^{r_1}u\,(r)J_0(\lambda_n r)rdr$$

Where λ_n are the eigenvalues defined by the Sturm-Liouville problem Because the boundary condition is of the Dirichlet type, the eigenvalues are defined as the roots of the following equation as can be found in Integrated Engineering Mathematics [1].

$$J_0(\lambda_\mathrm{n}\mathrm{r}_1)=0$$

The operational property of the Discrete Hankel Transform is defined as:

$$H\left\{\frac{1}{r}\frac{\partial}{\partial t}\left(r\frac{\partial u}{\partial t}\right)\right\}=r_1\lambda_\mathrm{n}J_1(\lambda_\mathrm{n}r_1)g(t)-\lambda_n^2\bar{u}_n$$

Using the operational property found in Integrated Engineering Mathematics [1], the equation is transformed to:

$$\frac{\partial^2 u}{\partial t^2}+\nu^2\lambda_n\bar{u}_n=\nu^2 r_1\lambda_n J_1(\lambda_n r_1)g(t)$$

With transformed initial conditions: $\bar{u}_0$ and $\bar{u}_1$

SOLVING LINEAR ODE

α is defined as:

$$\alpha=\nu^2\lambda_n r_1 J_1(\lambda_n r_1)$$

The transformed equation can be written as:

$$\frac{\partial^2 u}{\partial t^2}+\nu^2\lambda_n\bar{u}_n=\alpha g(t)$$

ω_n is defined as:

$$\omega_n=\sqrt{\nu^2\lambda_n}$$

The solution to the homogeneous transformed ODE is:

$$\bar{\mathrm{u}}_{\mathrm{hn}}=c_1 cos\left(\sqrt{\nu^2\lambda_n}t\right)+c_2 sin\left(\sqrt{\nu^2\lambda_n}t\right)$$

By Green's theorem for linear wave operators, the solution can be written as a convolution of the Green kernel with the forcing term[2].

$$\bar{u}_{pn}=\frac{\alpha}{\omega_\mathrm{n}}\int_0^t\sin\left(\omega_n(t-\tau)\right)g(\tau)d\tau$$

The solution is then defined as the superposition of the homogeneous solution and the particular solution:

$$\bar{u}_n=\bar{u}_{hn}+\bar{u}_{pn}$$

Using the inverse Hankel transform [1], the solution to the wave equation is:

$$\boxed{\sum_{n=1}^{\infty}\bar{u}_n\frac{J_0(\lambda_n r)}{\frac{r_1^2}{2}J_1(\lambda_n r_1)}}$$

IMPLEMENTING CONSTRAINTS

Plugging in T for t into the solution to the transformed ordinary differential equation and solving for the convolution integral yields:

$$\int_0^T\sin\left(\omega_n(t-\tau)\right)g(\tau)d\tau=S_n$$

Solving for the convolution integral for the time derivative of the same function results in:

$$\int_0^T \cos\left(\omega_n(t-\tau)\right)g(\tau)d\tau = C_n$$

Where S_n and C_n are defined as:

$$S_n = \frac{-\omega_n}{\alpha}\left(\bar{u}_0\cos(\omega_n T) + \bar{u}_1\sin(\omega_n T)\right)$$

$$C_n = \frac{1}{\alpha}\left(\omega_n\bar{u}_1\cos(\omega_n T) - \omega_n\bar{u}_0\sin(\omega_n T)\right)$$

A piecewise constant ansatz can be used to approximate $g(t)$ because it maps linearly onto the governing equations:

$$g(t) \approx \sum_{k=1}^{M} \gamma_k\, \phi_k(t)$$

The integrals can be quantized into the same bins as $g(t)$:

$$I_s(n,k) := \int_{t_k}^{t_{k+1}} \sin\left(\omega_n(t-\tau)\right)d\tau$$

$$I_c(n,k) := \int_{t_k}^{t_{k+1}} \cos\left(\omega_n(t-\tau)\right)d\tau$$

Substituting this into the equations:

$$S_n \approx \sum_{k=1}^{M} \gamma_k\, \phi_k I_s(n,k)$$

$$C_n \approx \sum_{k=1}^{M} \gamma_k\, \phi_k I_c(n,k)$$

To solve for the coefficients of the piecewise constant ansatz, a linear equation is defined:

$$\vec{g} = A^{-1}\vec{b}$$

Where $\vec{g}$ are the coefficients of the piecewise constant ansatz and A is defined as:

$$A = \begin{bmatrix} I_s(1,1) & I_s(1,2) & \dots & I_s(1,M) \\ I_c(1,1) & I_c(1,2) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ I_s(n,1) & I_s(n,2) & \dots & I_s(n,M) \\ I_c(n,1) & I_c(n,2) & \dots & I_c(n,M) \end{bmatrix}$$

And $\vec{b}$ is defined as:

$$\vec{b} = \begin{bmatrix} S_1 \\ C_1 \\ \vdots \\ S_n \\ C_n \end{bmatrix}$$

Solving this problem yields the solution for g such that the displacement and velocity of the membrane will go to zero at time T.

RESULTS AND DISCUSSION

The boundary controlling function, g(t), consistently succeeds in cancelling the membrane motion at the final time, T, driving both displacement and velocity to zero. This holds to be true for a variety of initial conditions and confirms that the transform-based approach of solving this problem correctly captures the governing physical dynamics.

We find that the problem was formulated in a way that makes it hard to connect to a real-world scenario. The boundary does not act by gradually suppressing the wave amplitude by matching the displacement in a direct or intuitive way. Instead, the boundary acts as what we call "the mode hunter", entering the system and moving in a way that isolates and destroys the eigenmodes one by one. This of course leads to the vanishing of the overall wave field at the prescribed time, even though the interior membrane motion may not decrease over time. The boundary is not simply reducing the amplitude of the wave everywhere at once but rather responding dynamically to the structure of the solution. By following the membrane motion as it reaches the boundary and injecting the proper boundary displacement, the control function eliminates the contribution of each eigenmode as it arrives.

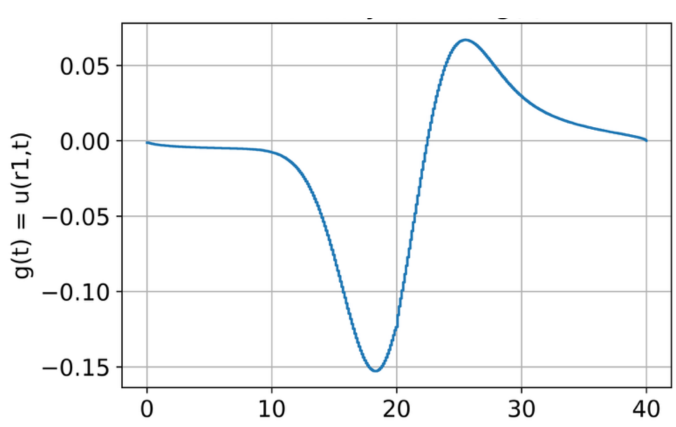

Figure 1: Boundary control of the wave equation with an initial Gaussian pulse.

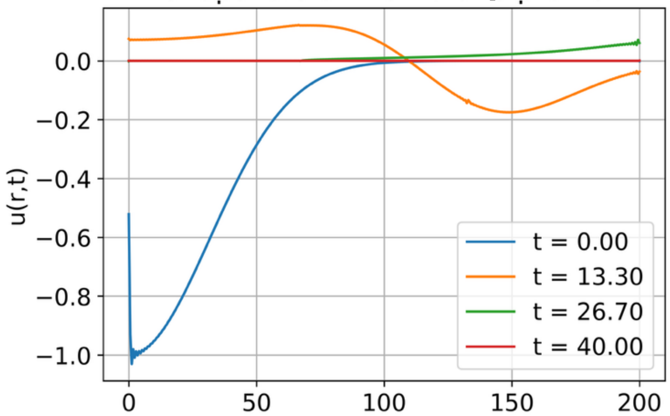

Figure 2: Membrane cross-section at various time steps

Figures 1 and 2 illustrate this behavior clearly. The boundary control function, g(t), evolves periodically, closely following the membrane motion at the boundary rather than enforcing a measured and out-of-phase destructive wave. At the same time, boundary-generated waves propagate inward and interact with the initial disturbance, gradually reshaping the membrane response. Despite all the dynamic motion of the membrane, the net effect is the steady removal of wave motion, and at the final time T, both the displacement and velocity of the membrane vanish throughout the domain, confirming that the boundary-driven control successfully achieves wave cancellation.

Initial conditions change the subsequent propagation of waves, which means that the initial conditions also change the control input required to destroy waves. This can be seen in figures 3 and 4, where the same Gaussian pulse as figures 1 and 2 was applied, except on the initial velocity rather than the initial displacement. As can be seen, this change led to drastic differences in the boundary control and waves that propagated. Each of the wave parameters seems to change the boundary control, leading to what seems to be a unique boundary control per set of parameters and initial conditions.

Together, these results show that successful wave cancellation in this system is achieved through proper timing and coordination at the boundary rather than through continuous suppression of the wave signal field. The boundary displacement adapts to the incoming motion and responds in such a way that it removes the wave's eigenmodes one by one. Although the way the membrane evolves appears to be complex before the final time, the outcome is consistent and repeatable: the boundary control drives the system to rest.

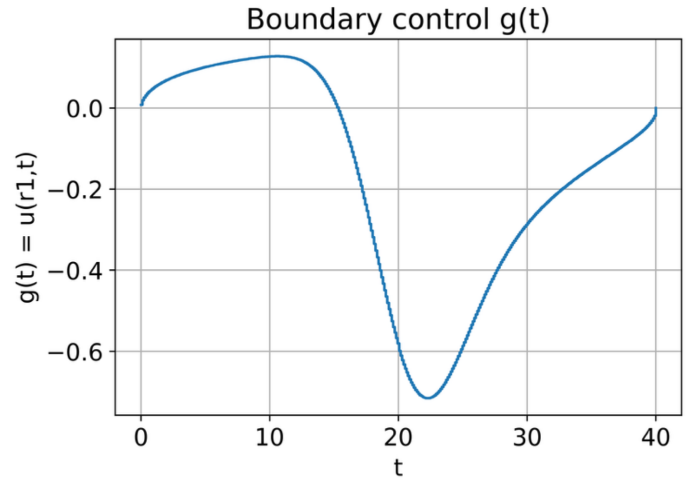

Figure 3: Boundary Control for a Gaussian Pulse of the Initial Velocity

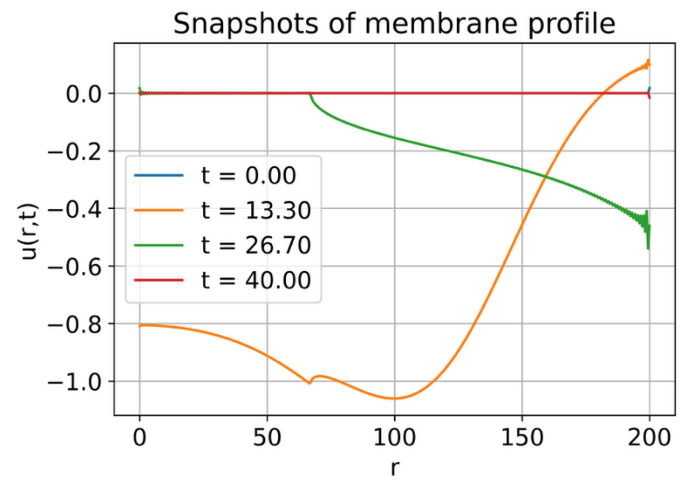

Figure 4: Membrane cross-section of a Gaussian pulse of the initial velocity.

CONCLUSIONS

Our project was an attempt at applying active noise cancellation through boundary control to a radially propagating wave on a circular membrane. After establishing the wave equation and its initial conditions, we applied the Hankel transform and the appropriate Sturm-Liouville eigenfunction problem in order to express our solution in terms of the Bessel equation, noting its eigenmodes. We then applied the inverse transform, which led us to utilize a piecewise-constant function for the boundary control, g(t). This led to the production of a linear system whose solutions cancel the outgoing waves. This approach shows how integral transform methods and constraint-based control techniques can provide an effective solution for boundary control of radial wave propagation.

Future work could include solving for a forcing term to achieve similar results. In practice, similar boundary-driven control strategies could be implemented using active boundaries, such as distributed actuators or speaker arrays, to cancel vibrations or acoustic waves in membranes or thin structures. This could result in better noise cancellation in enclosed spaces or better fatigue reduction in components subject to amplifying vibration modes. Similarly, using a more physically realistic boundary condition, such as the Robin conditions, would lead to more physically representative results.

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REFERENCES

[1] Solovjov, V., "INTEGRATED ENGINEERING MATHEMATICS."
[2] Strauss, W. A., 2008, *Partial Differential Equations: An Introduction*, Wiley, Hoboken, NJ.

