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# USING THE 1D WAVE EQUATION TO INFORM GUITAR DESIGN

**Alexandra Comer*, Hollis Waites†**

* Electrical Engineering Department, † Mechanical Engineering Department
Brigham Young University
Provo, Utah 84602
* agcomer@byu.edu, † hollish@byu.edu

## ABSTRACT

The 1D wave equation is a useful model for depicting guitar string vibrations. However, most descriptions do not break the solution down into common guitarist terms. Understanding this relationship better could improve students' and musicians' understandings of both the wave equation and guitar design. To bridge this gap, a model is derived for the displacement of a guitar string after a strumming or plucking input. Observations are made on the relationship between the variables, musical terms, and the guitar's sound. Finally, the model is compared to the performance of a real guitar.

## NOMENCLATURE

| | |
|---|---|
| x | Distance along the length of the string |
| L | The length of the string, bridge to nut |
| t | Time |
| u(x,t) | Displacement of the string |
| $\nu$ | Wave equation constant |
| $F_T$ | Tension force in the string |
| μ | The mass per unit length of the string |
| g | Gage thickness of the string (*in*) |
| ρ | Average density of the string |
| fret | A position on the length of the guitar neck |

## INTRODUCTION

Math models can be useful for designing and predicting the performance of physical products. The wave equation has been used to model the real life performance of stringed instruments, such as a guitar in simplified [1-3] or more detailed models. Knowledge of how math models relate to the performance of musical instruments can improve students' understanding of both the underlying math and the instrument itself. However, many of these derivations focus on the mathematical expressions, without connecting them directly to variables used by musicians or reported by instrument manufacturers.

This work seeks to bridge the gap between the higher level math and the practical understanding of musicians. First, this work derives a simple model for the deflection of a guitar string using the 1D wave equation, similar to the work of [1,3]. The constants in the wave equation are then translated to commonly listed guitar string values and conclusions are made about how each of these constants affects the guitar's sound. The model performance is then compared to that of a real guitar to evaluate the accuracy of the model.

## METHODOLOGY

To determine how the design of a guitar relates to its musical performance, first a simplified model is derived for the string's displacement over time based on the 1D wave equation. Then, performance behavior of an actual guitar is calculated to determine the accuracy of the model.

To model the vibrations of a guitar string, the 1D wave equation is used. Here, u(x,t) represents the displacement of the string, u, along its length, x, at time, t. The constant, $v$, is assumed to be a constant and will be discussed later.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

For a guitar string, pinned at either end, the boundary conditions can be simplified to:

$$u(0,t) = 0$$
$$u(L,t) = 0$$

For a piecewise input, f(x), representing the guitar being plucked at the beginning of the simulation, the initial conditions can be expressed as:

$$u_0(x,0) = f(x)$$
$$\frac{\partial u(x,0)}{\partial t} = 0$$

This partial differential equation is separable. To solve, we assume the product solution.

$$u(x,y) = X(x)T(t)$$

These variables are then applied to the original PDE and solved by separation of variables. Because either side of the equation is independent of the other, yet equal to each other, we know that it equals a constant, $\boldsymbol{\lambda}$, the eigenvalue.

$$\frac{X''}{X} = \frac{1}{v^2}\frac{T''}{T} = -\lambda$$

Now they are split into the spatial and temporal halves of the equation, and solve each equation for $\boldsymbol{\lambda}$. First, the spatial equation is:

$$X'' + \lambda X = 0,\ X(L) = 0,\ X(0) = 0$$

Its solution is known to be [cite the textbook table]:

$$X_n(x) = sin(\frac{n\pi x}{L}),\ \lambda_n = (\frac{n\pi}{L})^2$$

The temporal portion and its homogenous initial condition are:

$$T'' + \lambda v^2 T = 0,\ \frac{\partial u(x,0)}{\partial t} = 0$$

Its solution is also known to be [4]:

$$T_n(t) = A_n cos(\frac{n\pi v}{L}t) + B_n sin(\frac{n\pi v}{L}t)$$

Applying the initial condition, the temporal solution becomes:

$$T_n(t) = A_n cos(\frac{n\pi v}{L}t)$$

Combining the two parts of the product solution into one complete solution and summing all possible solutions (the eigenfunctions) into one complete solution, we have:

$$u(x,t) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L})cos(\frac{n\pi v}{L}t)$$

This is the particular solution. $A_n$, the amplitude of each harmonic, can be solved for the prescribed input function as follows. For brevity, steps have been skipped, but more details can be found in [1].

$$A_n = \frac{\int_0^L X_n(x)f(x)dx}{\|X_n\|^2} = \frac{2}{L}\int_0^L sin(\frac{n\pi x}{L})f(x)dx$$
$$= \frac{2L^2 u_1 sin(n\pi x_1/L)}{x_1(L-x_1)\pi^2 n^2}$$

The input function for a guitar string being plucked or strummed, f(x), can be expressed as a piecewise function [1,3], with variables defined as shown in Figure 1.

$$f(x) = \frac{u_1}{x_1}x\ for\ 0 \le x \le x_1$$
$$f(x) = \frac{u_1}{L-x_1}(L - x)\ for\ x_1 < x \le L$$

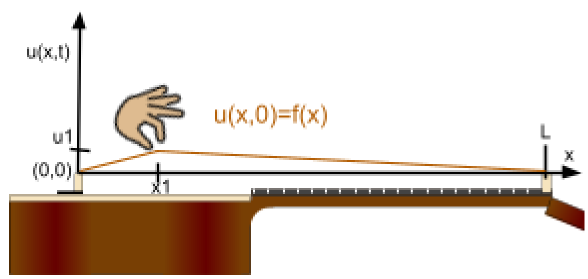


**Figure 1.** Model of the guitar string, demonstration of initial conditions, and basis for length measurement on actual guitar

The variable $v$, is known to be the speed of the wave. In a string, this can be calculated using the following equation: $\sqrt{F_T/\mu}$ [1], where $F_T$. is the tension of the guitar string in Newtons and μ is mass per unit length in kg/m. The variable μ can be calculated by finding the cross-sectional area of the string from its gage, g, (diameter) of the string and ρ, the density of the string material. This modifies the particular solution like so:

 

$$u(x,t) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L})cos(\frac{n\pi}{L}\sqrt{\frac{4F_T}{\pi g^2 \rho}}t)$$

By definition: the harmonic (sine or cosine) term, which is a function of time, contains constants which relate to the resonant frequencies of each harmonic, given as $f_n$.

$$f_n = \frac{n\sqrt{4F_T}}{2Lg\sqrt{\rho\pi}}$$

To determine the accuracy of this model, several variables are measured on an actual guitar, as shown in Figure 2. The manufacturer's listed gauge (string thickness) and material (string density) are used to determine μ, the 1D density of the string. The tension force, $F_T$, was experimentally measured using the method described by P. Ruiz [5]. After applying an input at distance x, the frequency of the guitar strings were measured using the "Plot Spectrum" function in Audacity on a '.mp3' file of each open string being plucked twice.

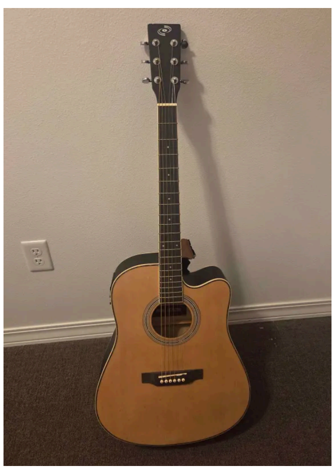

**Figure 2.** Guitar used to gather values

## DISCUSSION AND RESULTS

Several observations can be made from the amplitude equation and the restatement of the resonant frequency and deflection equation in common guitar terms. First, it is worth nothing that the string thickness and material affect only the resonant frequencies and not the amplitude. The first resonant frequency (n=1) corresponds to the musical pitch, or note, that is played. Increasing the string tension, decreasing the string gauge, or switching to a lower density string material all decrease the frequency, or pitch, of the note played. The volume of this note, corresponding to the amplitude $A_{n=1}$, increases with increasing string length (longer guitar necks), stronger strokes (larger initial deflection, $u_1$), and playing closer to the base of the guitar (decreasing $x_1$). The succeeding resonant frequencies, n > 1, correspond to other parts of the sound, such as 'warmth' or 'brightness' [6].

Additionally, although L was represented as the maximum length of the string, holding down the string at one of the frets on the guitar neck artificially decreases L to the position of that fret. Decreasing L in this way increases the pitch, or frequency, and increases the volume, or amplitude, of the note.

This model was tested experimentally. The experimentally measured frequency is tabulated in Table 1, along with the frequency calculated based on the resonant frequency equation, $f_n$.

**Table 1.** Comparison between experimental frequencies and calculated frequencies

| String | E | A | D | G | B | E |
|---|---|---|---|---|---|---|
| Gauge, g | 0.053 | 0.042 | 0.032 | 0.024 | 0.016 | 0.012 |
| Estimated Tension, $F_T$ (N) | 383 | 351 | 318 | 223 | 191 | 159 |
| String length, L (mm) | 650 | 650 | 650 | 650 | 650 | 650 |
| Experi-mental Freq (Hz) | 166 | 166 | 142 | 200 | 200 | 333 |
| Calculated Freq, $f_1$ (Hz) | 143 | 173 | 216 | 241 | 334 | 407 |

The calculated values in Table 1 followed the overall trend of the experimental values. However, there were large differences between the two (5-50% difference). Many of the variables used in the calculation were imprecise. For example, the density of the string depends on the gauge of the string, but a static value was used. Additionally, the measurement method for string tension required estimating the deflection of the string when pulled with a certain force. This deflection was difficult to measure and was mainly estimated by eyesight. Another evidence that the estimated tension was inaccurate is that recommended guitar tensions are typically much less than the values estimated in this work. This model also did not take into account the effect of other variables, such as the vibration of the guitar body or damping in the string, which may have affected the calculated frequency.

## CONCLUSIONS

The 1D wave equation was derived in detail for the conditions of a guitar string being plucked. This mathematical model was also tested against a real guitar's performance. There were many imprecisions in the experimental versus calculated values. In the future, it would be beneficial for the topic to be reinvestigated when more data is available. Additional errors could come from the recording device or Audacity. Furthermore, additional terms could be added to the equation. This would account for the rigidity of the steel strings among other real life factors, which are often ignored in calculations, that have an impact on the utility of a guitar.

Translating the solution into guitar terms yielded many useful conclusions. First, the choice of guitar string affects the pitch and other resonant frequencies, but not its volume. The pitch was also affected by the string tension and the string length (defined by either the length of the guitar's neck or by which fret the string is held at). The amplitude was only affected by the string length and the distance and depth of the input strum. This matches observations commonly known to guitarists and confirms the usefulness of this model. This solution of the 1D wave equation for a guitar string, expressed as common guitar terms, can provide quick and powerful understanding of both upper level math and guitar design choices to students and musicians alike.

## ACKNOWLEDGMENTS

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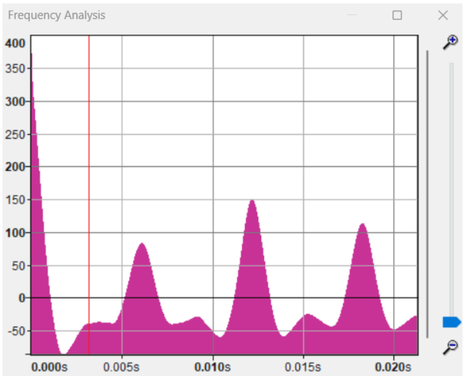
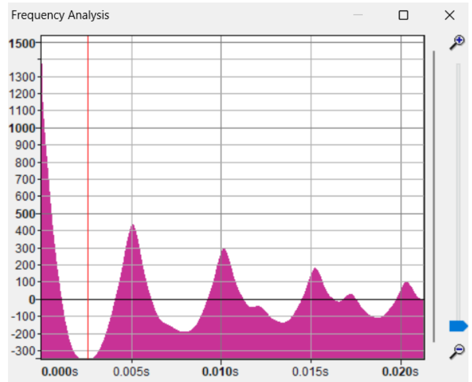
## APPENDIX A: Experimental waveforms

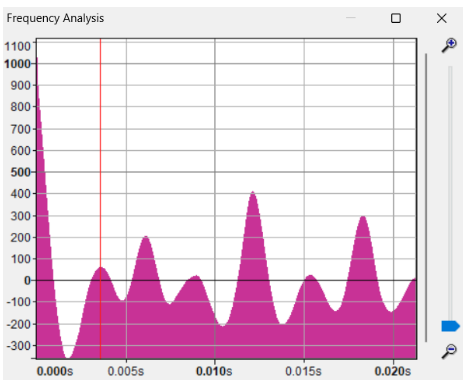
String E: (166 Hz)



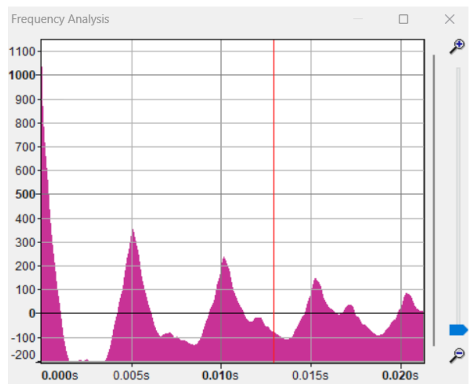

String G: (200 Hz)



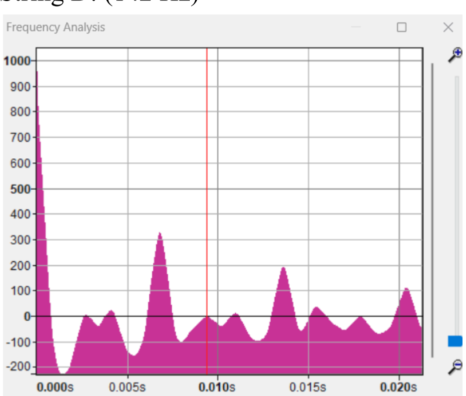

String A: (166 Hz)




String B: (200 Hz)




String D: (142 Hz)




String E: (333 Hz)