APPORXXIMATING DEFORMATION OF INFLATABLE TUBULES USING POISSON’S EQUATION

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ABSTRACT

This paper presents a method to model the displacement of an inflated laminate by solving Poisson’s equation in two dimensions. Other methods have analyzed the volume of an inflated laminate (teabag problem), but an analytical solution for the surface profile has not been found. The solution presented in this paper analyzes a rectangular laminate. The resulting solution is compared to physical testing. While the solution is inaccurate at some areas near the boundary, the calculated solution closely approximates the deformation along the center cross-section. The solution of the Poisson’s equation allows for a quick starting point in initial analysis.

NOMENCLATURE

controlled buckling
antagonistic shrinking

INTRODUCTION

A team at the Compliant Mechanisms Research group of BYU recently developed a unique, proprietary diaper pad. The pad was designed to reduce sag in diapers and allow for better conformance to the body when wetted. The diaper pad is a laminate of two absorbent cloth sheets sandwiching a layer of super absorbent polymer (SAP). These sheets are sewn together in a series of curved stitches. When the laminate comes in contact with water, the SAP expands and exerts uniform pressure on the top and bottom sheets. The result of this SAP expansion is a series of curved tubules whose geometry is analogous to circular origami pleating and is related to Salkowski curves. Once the laminate is fully liquid activated, the result is the unique doubly curved surface seen in Fig. 1b.

Figure 1. a) Flat diaper pattern before liquid activation, and b) doubly curved diaper pad post-activation.

This genre of laminate can assume various tubule patterns which produce interesting and potentially useful shapes (see Fig. 2). Fig. 2c shows a modified herringbone pattern with two tubules along the length of the pattern. This configuration creates antagonistic shrinking where the lengthwise tubules meet the shorter angled tubules, resulting in the buckling seen in Fig. 2d. This antagonistic shrinking has inspired several additional designs that consist of straight tubules arranged antagonistically to each other. An analysis of one these tubules—let alone a complex system of them—has yet to be performed. As such, this paper attempts to analytically approximate the shape of a single straight tubule arising from a rectangular laminate. This is accomplished using Poisson’s equation in two dimensions.

Figure 2. a) Additional example of curved tubules, b) circular tubule hypar, c) modified herringbone pattern, d) modified herringbone pattern post-activation.
BACKGROUND

The “teabag” problem attempts to answer the following question: if two inextensible square surfaces are joined at their edges, what is the maximum volume that it can hold when inflated? The solution to this problem has been approximated [1, 2] and a general case for the volume of a rectangular pillow under the same constraints has been developed as well [3]. An analytical solution of the surface, however, has not been reported. Several authors have developed an analytical solution that accurately describes the cross-section of an axisymmetric inextensible membrane, one that starts from a flat circular boundary joined at the edges [4, 5, 6]. One author has even performed an FEA model that would be identical to the conditions presented in this paper [7]. None of these authors, however, have produced an analysis of the surface resulting from initially rectangular boundaries. The solution presented here approximates the surface generated at the short ends of an initially rectangular inflated membrane. This analysis, while notably incomplete, could provide the groundwork to develop a more robust analytical model of the buckling phenomena occurring at short ends of initially rectangular inflated membranes.

CONSTRUCTION OF PHYSICAL TUBULE

The tubule examined in this paper was created by sandwiching SAP in between two rectangular sections of absorbent cloth. The initially rectangular laminate can be seen in Fig. 3. An inch-grid tracing fabric was used to define a 2-inch by 10-inch rectangular boundary. The tubule is then immersed and inflates as the SAP absorbs the liquid. The tubule inflates until the SAP pressure reaches equilibrium with the tensile stresses of the fabric.

PARTIAL DIFFERENTIAL EQUATION FORMULATION

Poisson’s equation in two dimensions describes a stationary boundary value partial differential equation that can be solved with the method of separation of variables. While in the case of a single tubule the boundaries change due to buckling, in certain patterns of multiple structures, the tubule boundary is constrained to a rectangle.

From early experimental results and due to symmetry, the problem can be simplified to only analyze a quarter of the full tubule structure. Knowing that the center of the tubule has a circular cross-section, one boundary value can be modeled by a semi-circle. The other three sides of the rectangular model can be represented by homogeneous Dirichlet boundaries. The pressure of the SAP is modeled by a semi-circular forcing function along the length of the structure. This is in attempt to more accurately simulate the pressure in the physical structure. Fig. 4 shows a diagram describing the setup of the Poisson’s equation for these conditions.

\[ \nabla^2 u + F = 0 \]

where

\[ L = 2 \]  
\[ M = 5 \]  
\[ f_1(x) = \sqrt{\frac{L^2}{\pi} - \left( x - \frac{L}{2} \right)^2} \]  
\[ F(x) = 2.85 \sqrt{\frac{L^2}{\pi} - \left( x - \frac{L}{2} \right)^2} \]

The coefficient in the forcing function was chosen to approximate the final shape of the tubule. As the mass of the material and the exact pressure is not considered in this model, this value was chosen experimentally.

COMPARISON OF SOLUTIONS

By using the method of separation of variables and superposition, the solution of the PDE can be represented by the following equations.

\[ u(x, y) = u_1 + u_2 \]  
\[ u_1 = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \sinh(\lambda_n y) \]
\[
\lambda_n = \frac{n\pi}{L} \text{ for } n = 1, 2, \ldots (7)
\]

\[
a_n = \frac{L}{2\sinh(-M \times \lambda_n)} \left( \int_0^\infty f_k(x) \sin(\lambda_n x) \, dx \right)
\]  

\[
u_m = \frac{n\pi}{M} \text{ for } n = 1, 2, \ldots (10)
\]

\[
A_{nm} = \frac{1}{(\lambda_n^2 + \nu_m^2) \times \frac{M}{2} \times \frac{L}{2}} \left( \int_0^M \int_0^L F \sin(\lambda_n x) \sin(\nu_m y) \, dx \, dy \right)
\]  

The resulting surface is seen in Fig. 5. When compared to the physical results, the profiles are very similar. To more closely match the boundary condition, the prototype was pulled until the short boundary was nearly flat in Fig. 5 and Fig. 7.

Table 1 shows a comparison of several values along the center cross-section in the \( y \)-direction of both the mathematical and physical models.

<table>
<thead>
<tr>
<th>Position (x,y) [in.]</th>
<th>Physical Height [in.]</th>
<th>PDE Solution [in.]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,4.5)</td>
<td>.40</td>
<td>.365</td>
<td>9.4%</td>
</tr>
<tr>
<td>(1,4.0)</td>
<td>.50</td>
<td>.526</td>
<td>5.2%</td>
</tr>
<tr>
<td>(1,3.5)</td>
<td>.58</td>
<td>.598</td>
<td>3.0%</td>
</tr>
<tr>
<td>(1,3.0)</td>
<td>.61</td>
<td>.628</td>
<td>3.0%</td>
</tr>
<tr>
<td>(1,2.0)</td>
<td>.62</td>
<td>.628</td>
<td>1.3%</td>
</tr>
</tbody>
</table>

This comparison shows that on the centerline along the length of the tubule, the model becomes more accurate approaching the center of the structure \((y = 0)\). Figures 6-8 show several different profiles of both the physical and mathematical model.

**DISCUSSION OF RESULTS**

As expected, the solution at the boundaries \( x = 0 \), and \( x = L \) differs from the actual physical solution. The buckling of these boundaries is caused by the inextensible nature of the material used to create the tubule. As the pressure increases, the boundaries change shape. While this model does not capture that...
deformation, it offers a reasonably accurate representation of the shape of the tubule closer to the center of the model.

In order to increase the fidelity of this model, the boundaries could be represented by parametric functions instead of straight rectangular boundaries. A different forcing function that varies along the length of the tubule may also yield more accurate results.

The most accurate results are still achieved employing a numerical or finite element approach as seen in previous work. These methods can determine the stresses felt in the material and find intermediate states as the pressure increases. However, the highly non-linear behavior, large displacements, complicated material properties, and computational resources make these solutions impractical for fast approximations. The solution presented here offers a rapid approximation that allows for a baseline understanding of the structure of a tubule.

CONCLUSIONS

This paper presents a solution to Poisson’s equation in two dimensions to model the structure of an inflated tubule. While the approach presented uses rectangular boundaries, using parameterized boundaries can be used for different configurations. The resulting solution offers a simple approximation of the displacement of the cloth. For more accurate simulations, finite element analysis can be used to solve for the displacement and stresses experienced by the cloth.

To solve the problem using finite elements, a material model for the cloth would need to be generated with consideration for any hyperelastic, non-linear, or laminate behavior. The swell pressure generated from the SAP would also have to be measured, and anisotropy of any fabric used would have to be considered. These more complex models could help predict the behavior of the tubules in different patterns or help predict if and how the structure may fail.

ACKNOWLEDGMENTS

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REFERENCES

APPENDIX-MATHEMATICA CODE

Poisson’s Equation Solution of an Inflated Tubule

◼ Establish boundary conditions and forcing functions

\[
L = 2; \\
M = 5; \\
n = 50; \\
\text{Low} = \frac{L}{2} - \frac{L}{\pi}; \\
\text{High} = \frac{L}{2} + \frac{L}{\pi}; \\
f = \text{Piecewise}[\{\text{Sqrt}[\left(\frac{L}{\pi}\right)^2 - (x - \frac{L}{2})^2], \text{Low} < x < \text{High}\}, 0]; \\
F_{\text{big}} = \text{Piecewise}[\{\text{Sqrt}[\left(\frac{L}{\pi}\right)^2 - (x - \frac{L}{2})^2], \text{Low} < x < \text{High}\}, 0];
\]

◼ Solution of Laplace Eq.

\[
\lambda_n = \text{ConstantArray}[0, n]; \\
\text{Do}[\lambda_n[i] = i \cdot \frac{\pi}{L}, \{i, 1, n\}]; \\
X_n[\lambda_] := \sin[\lambda \cdot x]; \\
Y_{n1}[\lambda_] := \sinh[\lambda \cdot (y - M)]; \\
an1 = \text{ConstantArray}[0, n]; \\
[\text{Do}[an1[i] = (1 / \sin[\lambda_n[i]] - M)] * N[\text{Integrate}[f \cdot \sin[\lambda_n[i]] \cdot x] \cdot (2 / L), \{x, 0, L\}], \{i, 1, n\}]; \\
u_1 = \text{Sum}[an1[i] \cdot X_n[\lambda_n[i]] \cdot Y_{n1}[\lambda_n[i]]], \{i, 1, n\}];
\]

◼ Solution of Poisson Eq with homogeneous boundaries

\[
\nu_m = \text{ConstantArray}[0, n]; \\
\text{Do}[\nu_m[i] = i \cdot \frac{\pi}{M}, \{i, 1, n\}]; \\
X_n[\nu_] := \sin[\nu \cdot x]; \\
Y_{n2}[\nu_] := \sin[\nu \cdot y]; \\
anm = \text{ConstantArray}[0, \{n, n\}]; \\
\text{Do}[\text{Do}[anm[i, j] = N[\text{Integrate}[\(\text{an1}[i] \cdot \sin[\lambda_n[i]] \cdot (M) / (2 / L) \cdot \nu_2[\nu_m[j]] \cdot y, 0, M])]] \\
((\lambda_n[i]^2 + \nu_m[j]^2) \cdot (M / 2)), \{i, 1, n\}, \{j, 1, n\}]; \\
u_2 = \text{Sum}[\text{Sum}[anm[i, j] \cdot X_n[\lambda_n[i]] \cdot Y_{n2}[\nu_m[j]]], \{j, 1, n\}], \{i, 1, n\}]; \\
u = u_1 + 2.85 \cdot u_2; \\
\text{Plot3D}[u, \{x, 0, L\}, \{y, 0, M\}, \text{PlotRange} \to \text{All}]
\]